AMPLE VECTOR BUNDLES ON ALGEBRAIC SURFACES

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The positivity of the Chern classes $c_i$ of an ample vector bundle on an algebraic surface is studied. Notably the inequality $0 < c_2 < c_1^2$ is established. This inequality was conjectured by Hartshorne [5] and Griffiths [1] (for compact, complex manifolds).

Let $X$ be a scheme of finite type over an algebraically closed field, $E$ a vector bundle on $X$ (i.e., a locally free sheaf of constant, finite rank), and $S^n(E)$ the $n$th symmetric power of $E$.

**Definition (Hartshorne [5])** 1. The bundle $E$ is ample if for every coherent sheaf $F$ on $X$, there is an integer $N > 0$, such that for every $n \geq N$, the sheaf $F \otimes S^n(E)$ is generated by its global sections.

**Proposition (Hartshorne [5])** 2. Consider the following conditions:

(i) The bundle $E$ is ample.

(ii) Let $P = \mathbb{P}(E)$ be the associated projective bundle and $L = \mathcal{O}_P(1)$ the tautological line bundle. Then $L$ is ample on $P$.

(iii) For every coherent sheaf $F$ on $X$, there exists an integer $N > 0$, such that for $n = N$ and $q \geq 1$

$$H^0(X, F \otimes S^n(E)) = 0.$$  

Then (i) and (ii) are equivalent and they are implied by (iii). If further, $X$ is complete, then (i), (ii) and (iii) are all equivalent.

**Theorem 3.** Let $X$ be an irreducible, nonsingular surface which is projective over an algebraically closed field, and let $A(X)$ be the Chow $\mathbb{R}$-algebra of cycles modulo numerical equivalence. Let $E$ be a vector bundle of rank $r \geq 2$ on $X$, and let $c_1, c_2 \in A(X)$ be the Chern classes of $E$. Assume $E$ is ample. Then, $c_2 > 0$ and $c_1^2 - c_2 > 0$.

**Proof.** Since $E$ is ample on $X$, then $\mathcal{O}_P(1)$ is ample on $P = \mathbb{P}(E)$. Hence, by [EGA II, 4.4.1, 4.4.2 and 4.4.10], there exist an integer $n \geq 2$ and a projective embedding, $j: P \to Y = \mathbb{P}^r$ such that $\mathcal{O}_P(n) = j^*\mathcal{O}_Y(1)$. For this embedding, let $S$ be the Chow variety parametrizing the 2-dimensional sections of $P$ by linear spaces and $T$ the subvariety of $S$ corresponding to those sections which meet a given fiber of $P \to X$ in infinitely many points. As $n \geq 2$, the codimension of $T$ in $S$ is at least 3.

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Let $H$ be a general 2-dimensional linear section of $P$. By the principal of counting constants, the map $H \rightarrow X$ has finite fibers; so, it is finite by [EGA III, 4.4.2]. Further, $H$ is irreducible and nonsingular by Bertini's theorems [EGA V].

Let $l$ be the class of $O_P(1)$ in the Chow algebra $A(P)$. By [2] or [3.1], $A(P)$ is generated over $A(X)$ by $l$ modulo the relation,

$$l^r - c_1l^{r-1} + c_2l^{r-2} = 0. \tag{3.1}$$

Let $a \in A^1(X)$. Then, $(l - c_1) \cdot l^{r-1} \cdot a = -c_2 \cdot a \cdot l^{r-2} = 0$. Let $h \in A(P)$ be the class of $H$; then, $h = (nl)^{r-1}$. Therefore,

$$(l - c_1) \cdot a \cdot h = 0. \tag{3.2}$$

Let $i: H \rightarrow P$ be the inclusion map and $i_*, i^*$ the maps induced on the Chow algebras. Then, $i_*i^*b = b \cdot h$ for $b \in A(P)$. In view of (3.2), it follows that for any $a \in A^1(X)$,

$$i^*(l - c_1) \cdot a = 0. \tag{3.3}$$

The Lefschetz hyperplane theorem [3.2, XIII, 4.6 (iii)$\Rightarrow$(vi)] implies that $i^*(l - c_1) \neq 0$ because $(l - c_1) \neq 0$. Let $a \in A^1(X)$ be the class of an ample line bundle. Since $H$ is finite over $X$, then $a \cdot 1_H \in A^1(H)$ is the class of an ample line bundle by [EGA II, 5.1.12]. In view of (3.3), the Hodge index theorem [3.2, XIII, 7.1] asserts that $0 > i^*(l - c_1)^2$; thence, by (3.1) and (3.3) with $a = c_1$, it follows that $0 > -c_2 \cdot 1_H$.

Similarly, since $i^*l$ is the class of an ample line bundle on $H$, then $0 < i^*l^2$; thence, by (3.1) and (3.2) with $a = c_1$, it follows that $0 < (c_2^2 - c_2c_1) \cdot 1_H$.

**Remark 4.** With the more general theory of Chern classes developed in [3.1], the same reasoning establishes that $c_2 > 0$ and $c_2^2 - c_2 > 0$ for an ample bundle $E$ on an arbitrary surface $X$. Consequently, on a projective algebraic scheme $Y$ of arbitrary dimension, an ample bundle $E$ has classes $c_2$ and $c_2^2 - c_2$ which have positive intersection number with every surface $X$ on $Y$.

**Example 5.** Under the conditions of Theorem 3, the inequality $c_2^2 - c_2 > 0$ is best possible in the following sense. There exists a sequence of ample, rank 2 bundles $E_n$ on $X$, such that for all $e > 0$, $(c_2^2 - (1+e)c_2(E_n))$ equals $-en^2d + \cdots$ with $d = \text{deg}(X)$, so it tends to $-\infty$ as $n \rightarrow \infty$.

To construct $E_n$, fix a surjection $\alpha_n: O_X(1) \rightarrow O_X(n)$. Let $F_n$ be the

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2 This line is due to Hartshorne who commented in private on the proof that $c_2 > 0$.
dual of the kernel of $\alpha_n$ and $E_n = F_n(1)$. Then $E_n$ is a rank 2 bundle and there is an exact sequence

$$0 \to O_X(1 - n) \to O_X(1) \oplus ^E \to E_n \to 0.$$ 

Hence, $c_1(E_n) = (n + 2)d$ and $c_2(E_n) = (n^2 + n + 1)d$. Finally, since $E_n$ is a quotient of a direct sum of ample line bundles, $E_n$ is ample [5, (2.2)].

In characteristic $p > 0$, there are two new notions extending the notion of ampleness for line bundles: Let $f: X \to X$ be the Frobenius ($p$th-power) endomorphism and $f^n$ the $n$th iterate of $f$.

**Definition (Hartshorne [5]) 6.** (i) The bundle $E$ is $p$-ample if for every coherent sheaf $F$, there is an integer $N > 0$, such that for every $n \geq N$, the sheaf $F \otimes f^n_* E$ is generated by its global sections.

(ii) The bundle $E$ is cohomologically $p$-ample if for every coherent sheaf $F$ on $X$, there is an integer $N > 0$, such that for $n \geq N$ and $q \geq 1$, $H^q(X, F \otimes f^n_* E) = 0$.

**Remark 7.** (i) Assume $X$ is quasi-projective. Then any coherent sheaf $F$ is a quotient of a sheaf of the form $O_X(-m) \otimes^M$ for $m, M > 0$. It follows that for the Definitions 6 (as well as for the analogous formulations of ampleness) it suffices to verify the condition on sheaves of the form $F = O_X(-m)$ for $m > 0$.

(ii) Hartshorne [5, (6.3)] proves that $p$-ample bundles are ample. He conjectures the converse, and proves it for line bundles and for curves [5, (7.3)].

**Example 8.** A $p$-ample bundle on a complete scheme need not be cohomologically $p$-ample. In fact, the rank 2 bundles $E_n$ constructed in (5) are $p$-ample being quotients of direct sums of $p$-ample bundles [5, (6.4)]; however, for $n \geq 2$, (although quotients of cohomologically $p$-ample bundles) they are not cohomologically $p$-ample because for $m \gg 0$, $H^1(X, f^n_* E_n)$ equals $H^2(X, O_X(-m(n-1)))$, which is $> 0$ by [6, p. 944].

**Proposition 9.** Suppose $X$ is quasi-projective and $E$ is cohomologically $p$-ample. Then $E$ is $p$-ample.

**Proof.** In view of (7)(i), fix an integer $m > 0$ and let $G_n = (f^n_* E)(-m)$.

Let $x \in X$ be a closed point. Then there is an $N$ such that the stalk $(G_N)_x$ is generated by global sections. Indeed, it suffices to show that the map $H^0(X, G_N) \to H^0(X, G_N \otimes k(x))$ is surjective. However, by hypothesis, there is an $N$ such that $H^1(X, I_x \otimes G_N) = 0$ where $I_x$ is the ideal defining $\{x\}$. There is, therefore, a neighborhood $U$ of $x$ in which $G_N$ is generated by global sections.
Let \( n = N + t \), with \( t \geq 0 \). Then, \((f^*_n E)(-mp') = f^*_n G_N\) is generated in \( U \) by global sections. However, for any sheaf \( G \) and \( r \geq 0 \), \( G \) is a quotient of \( G(-r)^{\otimes s} \) for suitable \( s \). Thus, \( G_n \) is generated in \( U \) by global sections. By quasi-compactness, it follows that \( E \) is \( p \)-ample.

**Lemma 10.** Suppose \( X \) is integral, quasi-projective and of dimension \( r \) and \( E \) is \( p \)-ample. Then for some \( a > 0 \),

\[
h^0(X, f^*_n E) \geq ap^n + \cdots .
\]

**Proof.** Take \( N \) such that \((f^*_n E)(-1)\) is generated by global sections. It follows that there is a map \( \beta : O_X(1) \to f^*_n E \) which is a split-injection on an open set. Let \( n = N + t \), with \( t \geq 0 \). Then, \( O_X \) being torsion free, \( f^*_n \beta : O_X(p^t) \to f^*_n E \) is an injection. Thus, \( h^0(X, f^*_n E) \geq h^0(X, O_X(p^t)) \); whence the conclusion.

**Theorem (Hironaka) 11.** Let \( X \) be an integral (nonsingular) surface which is projective over an algebraically closed field of characteristic \( p > 0 \), \( E \) a cohomologically \( p \)-ample bundle on \( X \), and \( c_1, c_2 \) the Chern classes of \( E \) modulo numerical equivalence. Then, \( c_1^2 - 2c_2 > 0 \).

**Proof.** For any bundle \( E \) on \( X \), the Riemann-Roch theorem implies that \( \chi(f^*_n E) = ((c_1^2 - 2c_2)/2!)p^{2n} + \cdots \). Suppose \( E \) is cohomologically \( p \)-ample. Then, in view of (9) and (10), \( E \) is \( p \)-ample and \( c_1^2 - 2c_2 > 0 \).

**References**

3.1. Exposes of Berthelot.
3.2. Expose of Kleiman.

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\(^1\) This result was in essence contained in a private communication from Hironaka.