A NOTE ON SQUARE INTEGRABLE SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS[1]

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The equation

(1) $(ry')' = qy$,

where r, q are locally integrable and $r(t) \neq 0$ is of limit point type on $(0, \infty)$ i.e. it has a solution which is not in $L^2(0, \infty)$ if

(i) $q \in L^2(0, \infty)$.

This result, for the case $r \equiv 1$, follows from a theorem of P. Hartman [**1**, §§1 and 2]. A simple proof of this special case, which can readily be adapted to yield the result mentioned above, is given in [**2**, Satz 7, p. 305].

In this note we show that the equation

(2) $Ly = qy$

where L has the form

(P) $Ly = r_n(r_{n-1} \cdots (r_1(r_0 y)')' \cdots)'$

with n even, q, r_i locally integrable and r_i having no zeros in $(0, \infty)$ for $i = 0, \cdots, n$ has a solution which is not in $L^2(0, \infty)$ if

(i)′ $q/r_n \in L^2(0, \infty)$, $1/r_0 \notin L^2(0, \infty)$ and

(ii) $r_i = r_{n-i}$ for $i = 1, \cdots, n$.

PROOF. Define $D_i y = r_i(D_{i-1}y)'$ for $i = 1, \cdots, n$ with $D_0 y = r_0 y$. Note that if y is an $L^2(0, \infty)$ solution of (2) then $D_{n-1}y$ is bounded since $(D_{n-1}y)t = \text{constant} + \int_0^t qy/r_n$. Let $y_1, \cdots, y_n$ be a set of linearly independent solutions of (2) and consider their generalized Wronskian

$$W = W(y_1, \cdots, y_n) = \det (D_{i-1}y_j).$$

Expanding W with respect to the elements of its last row we obtain

(3) $W = \sum_{i=1}^n Z_i D_{n-1} y_i$

where Z_i is the cofactor of $D_{n-1}y_i$. It can be established by direct computation that Z_i/r_0 are also solutions of (2). Hence $y_i \in L^2(0, \infty)$ for $i = 1, \cdots, n$ leads to the conclusion that the right side of (3) is in $L^2(0, \infty)$ which is a contradiction in view of the fact that W is a nonzero constant and hence $W/r_0 \notin L^2(0, \infty)$.

We conclude with some remarks.

REMARK 1. Our result above holds in particular for the class of operators L given by

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(4) $Ly=(ry^n)^n$

for r any nonvanishing locally integrable function. Note that no growth assumptions on r near infinity are needed. It follows [**2**, Satz 4, p. 203] from our result above that the equation

(5) $(ry^n)^n-qy=iy$

has less than $2n$ linearly independent $L^2(0, \infty)$ solutions. It would be interesting to know under what conditions the number of linearly independent $L^2(0, \infty)$ solutions of (5) is exactly n. For some results in this direction see [**2**, Chapter VII].

REMARK 2. G. Pólya [**3**] has shown that the classical operator

$$My = y^n + p_{n-1}y^{n-1} + \cdots + p_0y$$

can be factored into the form (P) if and only if the equation $My=0$ has a set $y_1, \cdots, y_{n-1}$ of linearly independent solutions satisfying

$$W_j(y_1, \cdots, y_j) = \det(y_j^{i-1}) > 0 \quad \text{for } j = 1, \cdots, n-1 \quad (W_1(y_1) = y_1).$$

REMARK 3. The proof given clearly remains valid if (i)′ is replaced by $q/r_n \in L^p(0, \infty)$, $1/r_0 \notin L^s(0, \infty)$ and the conclusion is modified to read: not all solutions of (2) are in $L^s(0, \infty)$ where $p^{-1}+s^{-1}=1$.

REFERENCES

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2. M. A. Neumark, *Lineare Differentialoperatoren*, Akademie-Verlag, Berlin, 1960.

3. G. Pólya, *On the mean-value theorem corresponding to a given linear differential equation*, Trans. Amer. Math. Soc. **24** (1922), 312–324.

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