A THEOREM ON THE SEMIGROUP OF BINARY RELATIONS

JAW-CHING YANG

The purpose of this note is to generalize a theorem of Zaretskii [2]. Notations and definitions used here are based on those of [1] and [2].

Theorem. Let $X$ be an arbitrary set. The necessary and sufficient condition that the binary relation $p$ is a regular element of the semigroup $S_X$ is that $L(p)$ is a completely distributive complete lattice.

Proof. Necessity. Let $p = p^2 p$, where $p \in S_X$. Let $\sigma = \delta p$, then $\sigma^2 = \sigma$ and $\rho = \rho \sigma$. It is known in [1] that $L(\sigma)$ and $L(p)$ are complete lattices in which joins are unions and, moreover, $L(\sigma)$ is completely distributive.

If $A \subseteq X$, then it is easy to show that $\psi(A) = \psi(\phi(A))$ and $\phi(A) = \chi(\psi(A))$ where $\phi(A) \in L(\sigma)$, $\psi(A) \in L(p)$ and $\chi(A) \in L(\delta)$. Define the mapping $\theta$ of $L(\sigma)$ onto $L(p)$ as follows: if $\phi(A) \in L(\sigma)$, then $\theta(\phi(A)) = \psi(A)$. Clearly, $\theta$ preserves set-inclusion order and is one-to-one. Hence, $L(\sigma)$ is completely isomorphic with $L(p)$. This proves that $L(p)$ is completely distributive.

Sufficiency. Let $L(p)$ be a completely distributive complete lattice. Define the binary relation $\delta$ as follows: $(x, y) \in \delta$, iff $\rho(x, y) \subset \rho$. Obviously, $\rho \delta p \subset \rho$.

For each $z \in X$, define $K_z = \{\psi(\{y\}) : z \in \psi(\{y\})\}$. For any $y \in X$, let $K = \{K_z : z \in \psi(\{y\})\}$ and $S(K)$ denote the set of mappings $s$ of $\psi(\{y\})$ into $L(p)$ such that for every $z \in \psi(\{y\})$, $s(z) \in K_z$. Then $\bigvee \{\bigwedge K_z : z \in \psi(\{y\})\} = \bigwedge \{\bigvee s(\psi(\{y\})) : s \in S(K)\}$. Since lattice joins are unions, we have $\bigvee s(\psi(\{y\})) \supseteq \psi(\{y\})$, for each $s \in S(K)$, and hence $\bigwedge \{\bigvee s(\psi(\{y\})) : s \in S(K)\} \supseteq \psi(\{y\})$. Therefore,

$$\bigcup \{\bigwedge K_z : z \in \psi(\{y\})\} = \bigwedge \{\bigvee s(\psi(\{y\})) : s \in S(K)\} \supseteq \psi(\{y\}).$$

Let $(x, y) \in \rho$. Then $x \in \psi(\{y\})$ and so there exists a $z \in \psi(\{y\})$ such that $x \in \bigwedge K_z$. Therefore $x \in \psi(\{z\})$ for some $w$ satisfying

$$\psi(\{w\}) \subseteq \bigwedge K_z \subseteq \bigcap K_z = \bigcap \{\psi(\{z\}) : z \in \psi(\{z\})\}.$$
But $w\delta z$ holds iff

$$\psi(\{w\}) \subseteq \bigcap \{\psi(\{z\}) : z \in \psi(\{v\})\}.$$ 

Thus, we have $x \rho w$, $w \delta z$ and $z \rho y$; hence $(x, y) \in \rho \delta \rho$. It follows that $\rho \subseteq \rho \delta \rho$. Therefore $\rho = \rho \delta \rho$. This completes the proof.

REFERENCES


2. K. A. Zaretskii, Regular elements of the semigroup of binary relations, Uspehi Mat. Nauk 17 (1962), 177–179. (Russian)

INSTITUTE OF MATHEMATICS, ACADEMIA SINICA, TAIWAN