ON PERIODIC MAPS WHICH RESPECT A SYMPLECTIC STRUCTURE

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The lacunae in our knowledge of the coefficients have made direct computations in the homology theory $\Omega^{Sp}_*(X, A)$ almost impossible. This note sidesteps the difficulty by exploiting the relationship between the theories $KO^*$ and $\Omega^{Sp}_*$, which corresponds to that between $KU^*$ and $\Omega^{P}_*$, and which is fully discussed in Chapter II of [1]. In this way we are able to calculate the additive orders of some important elements in $\Omega^{Sp}_*(\mathbb{Z}_p)$ ($p$ prime), the module of principal $\mathbb{Z}_p$-manifolds with compatible weak symplectic structure. Perhaps the most interesting aspect of the computation is the difference between the cases $p$ odd and $p$ equals 2, reflecting the presence of 2-torsion in $\Omega^{Sp}_*$.

I am grateful to Elmer Rees for discussing this problem with me and for showing me his version of the computations in [2] for real projective space.

Notation. Let $KU( )$, $KO( )$ and $KSp( )$ be the cohomology functors defined by unitary, orthogonal and symplectic bundles respectively. There are natural operators

$$KO(X) \xrightarrow{i} KU(X) \xrightarrow{j} KSp(X)$$

and complex conjugation $c: KU(X) \rightarrow KU(X)$. These satisfy the relations

$$pi = 2 \quad qj = 1 + c \quad pc = p$$
$$ip = 1 + c \quad jq = 2 \quad jc = j.$$

This notation will be used below without further comment.

The relation between $K$-theory and bordism. As in the SO- and U-bordism theories there is a natural isomorphism between $\Omega^{Sp}_*(\mathbb{Z}_p)$ and $\Omega^{Sp}_*(K(\mathbb{Z}_p, 1))$, defined by classifying a principal $\mathbb{Z}_p$-manifold as a principal $\mathbb{Z}_p$-bundle. Let $T$ be the weakly symplectic $\mathbb{Z}_p$-action which defines the lens space

$$L^{4n-1}(p; 1, \ldots, 1) = L^{4n-1}_p.$$  

$2n$ times

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When \( p \) equals 2, \( T \) reduces to the antipodal map \( A \). Under the isomorphism above \([T, S^{4n-1}]\) corresponds to the inclusion of \( L_p^{4n-1} \) in the infinite join, which is the standard \( K(\mathbb{Z}_p, 1) \). In [1] Conner and Floyd prove that the first cobordism Pontrjagin class \( p_1 : KSp(X) \to \Omega^4_{Sp}(X) \) is a monomorphism. If \( h_H \) is the canonical symplectic line bundle over \( HP(n) \), let \( \nu \in KSp(HP(n)) \) denote the class of \((h_H-1)\). The image of \( \nu \) under \( p_1 \) is the cobordism class of the inclusion \( u : HP(n) \hookrightarrow HP(\infty) \). Let \( \pi : L_p^{4n+3} \to HP(n) \) be the projection of the natural fibration, and consider the commutative diagram:

\[
\begin{array}{ccc}
\tilde{K}Sp(L_p^{4n+3}) & \xrightarrow{\pi^*} & 4n+3 \\
\uparrow p_1 & & \uparrow D \\
\tilde{K}Sp(HP(n)) & \xrightarrow{\pi^*} & \Omega^4_{Sp}(HP(n)).
\end{array}
\]

The isomorphism \( D \) on the right is Poincaré duality.

\[
p_1\pi^*\nu = \{u\pi : L_p^{4n+3} \to HP(\infty)\}.
\]

Regard \( HP(\infty) \) as a submanifold of codimension 4 in itself. Duality, which is a variant of the Thom representation theorem for bordism, then implies that the final image of \( \pi^*\nu \) is the bordism class \( \{L_p^{4n+3} \hookrightarrow L_p^{4n+3}\} \). Finally an application of the cellular approximation theorem shows that the map

\[
V_* : 4n-1(L_p^{4n+3}) \to \Omega^4_{Sp}(\mathbb{Z}_p)
\]

is also a monomorphism. This proves

**Lemma.** The order of \([T, S^{4n-1}]\) in \( \Omega^4_{4n-1}(\mathbb{Z}_p) \) (\( p \) any prime) is equal to the order of \( \pi^*\nu \) in \( \tilde{K}Sp(L_p^{4n+3}) \).

**Orders of the bordism classes.**

**Proposition 1.** For \( p \) odd \([T, S^{4n-1}]\) has additive order \( p^{k+1} \) in \( \Omega^4_{4n-1}(\mathbb{Z}_p) \), where \( k(p-1) \leq 2n+1 < (k+1)(p-1) \).

**Proof.** Let \( \pi_0 : L_p^{4n+3} \to CP(2n+1) \) be the natural fibration, and consider the composition

\[
L_p^{4n+3} \xrightarrow{\pi_0} CP(2n+1) \xrightarrow{\pi} HP(n).
\]

If \( h_C \) is the canonical line bundle over \( CP(2n+1) \), denote the class
of \((h_c-1)\) in \(\tilde{K}U(CP(2n+1))\) by \(\mu\), and the pullback of \(\mu\) along \(\pi_0\) by \(\sigma\). Since the operator \(j\) commutes with continuous maps, and the pullback of \(\nu\) is \(j\mu\),

\[\pi' = \pi' \circ_0 \mu = j\sigma.\]

The same argument as that used by T. Kambe in [3], only with \(\tilde{K}Sp\) replacing \(\tilde{K}O\), shows that \(\tilde{K}Sp(L^{4n+3}_p)\) consists of \(p\)-torsion. (The top dimensional cell cannot introduce a factor \(\mathbb{Z}_2\), since \(4n+3 \not\equiv 5 \text{ mod } 8\).) Hence for the pair of operators \((j, q)\), where \(jq\) is multiplication by \(2\), \(q\) is a monomorphism and \(j\) an epimorphism. Composing in the reverse order, \(qj(\sigma) = (1+c)\sigma\). From the properties of the complex line bundle over \(CP(2n+1)\), and the naturality of \(q\) with respect to \(\pi_0\), it follows that

\[q(j\sigma) = \sigma^2 - \sigma^3 + \sigma^4 - \cdots.\]

T. Kambe (op. cit.) has computed the group \(\tilde{K}U(L^{4n+3}_p)\), and it follows from his calculations that \(q(j\sigma)\), and hence \(jq\), have order \(p^{k+1}\), where \(k(p-1) \leq 2n+1 < (k+1)(p-1)\). Proposition 1 now follows from the Lemma.

The situation when \(p\) equals 2 is more complicated, since in the presence of 2-torsion it is not clear that \(j\) is an epimorphism. Also in this special case we replace the symbol \(T\) by \(A\).

**Proposition 2.** (i) \([A, S^{4n+1}]\) has order \(24n+3\) in \(\Omega^{Sp}_{4n+3}(\mathbb{Z}_2)\), and (ii) \([A, S^{4n+7}]\) has order \(24n+4\) in \(\Omega^{Sp}_{4n+7}(\mathbb{Z}_2)\).

**Proof.** The projection \(\pi\) reduces to the fibration of projective spaces \(RP(4n+3) \to HP(n)\) and \(h_H\) pulls back to the canonical line bundle \(h_R\) over \(RP(4n+3)\) regarded as a symplectic bundle. Denote the class of \((h_R-1)\) in \(\tilde{K}O(RP(4n+3))\) by \(\lambda\), when \(\pi' = ji(\lambda)\). It is not obvious that \(ji(\lambda)\) is always a generator of \(\tilde{K}Sp(RP(4n+3))\), but from Bott periodicity and the fact that \(\lambda\) generates \(\tilde{K}U(RP(4n+3))\), it follows that it is enough to look at the cokernel of the forgetful map \(p\). Consider the long exact sequence of groups associated with the pair \((U, O)\) having quotient homotopically equivalent to \(\tilde{K}O\).

\[
\cdots \to \tilde{K}O^4X \to \tilde{K}O^4X \to \tilde{K}U^4X \to \tilde{K}O^4X
\]

\[\cdots \to \tilde{K}O^4X \to \tilde{K}U^4X \to \cdots.\]

From the computations of M. Fujii we have the particular cases:
\[ X = \text{RP}(8n + 3), \quad \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}_2 + \mathbb{Z}_2 \rightarrow \mathbb{Z}_{2^{n+1}} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \]

\[ X = \text{RP}(8n + 7), \quad \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow . \]

In both cases the map \( p \) is an epimorphism. Since we are only interested in the magnitudes of the various groups, we can use periodicity to replace the sequence (*) with that associated with the pair \((U, Sp)\), provided we shift the dimensions by 4. It follows that \( j \) is also an epimorphism, and that \( ji(\lambda) \) generates the cyclic group \( KSp(\text{RP}(4n+3)) \). Proposition 2 now follows from the Lemma.

**Additional remarks.** At least when \( p \) is odd, it is possible to show that the submodule of \( \tilde{\Omega}_{\ast}^{Sp}(\mathbb{Z}_p) \) generated by \( \{ [T, S^{4n-1}], w^1 \} \) is proper. The spectral sequence for \( \tilde{\Omega}_{\ast}^{Sp}(\mathbb{Z}_p) \) has nonzero terms at the \( E_2 \) level only when \( s \equiv 0 \) mod 4, \( r \equiv 1 \) mod 2, and hence collapses. This is a consequence of the isomorphism of \( \tilde{\Omega}_{\ast}^{Sp} \otimes \mathbb{Z}[1/2] \) with \( \Omega_{\ast}^{SO} \otimes \mathbb{Z}[1/2] \) as \( \mathbb{Z}[1/2] \) polynomial algebras, [4]. It follows that the edge homomorphism from \( \Omega_{\ast}^{Sp}(\mathbb{Z}_p) \) to \( H_{\ast}(\mathbb{Z}_p, \mathbb{Z}) \) is an epimorphism, and that our construction has failed to pick up a complete set of module generators.

As in the oriented, and in contrast to the unitary theory, the structure of \( \Omega_{\ast}^{Sp}(\mathbb{Z}_2) \) does not follow as a special case from that of \( \tilde{\Omega}_{\ast}^{Sp}(\mathbb{Z}_p) \). The general pattern of Proposition 1 would suggest that \([A, S^{8n+3}]\) and \([A, S^{8n+7}]\) have orders \( 2^{4n+3} \) and \( 2^{4n+5} \) respectively. In the second case the predicted order differs by a factor of 2 from the actual order; no doubt this is a consequence of the 2-torsion structure of \( \Omega_{\ast}^{Sp} \) itself.

**References**


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