ON PERIODIC MAPS WHICH RESPECT A SYMPLECTIC STRUCTURE

C. B. THOMAS

The lacunae in our knowledge of the coefficients have made direct computations in the homology theory $\Omega_*^{\mathrm{Sp}}(X, A)$ almost impossible. This note sidesteps the difficulty by exploiting the relationship between the theories KO^* and Ω_{Sp}^*, which corresponds to that between KU^* and Ω_U^*, and which is fully discussed in Chapter II of [1]. In this way we are able to calculate the additive orders of some important elements in $\Omega_*^{\mathrm{Sp}}(\boldsymbol{Z}_p)$ (p prime), the module of principal $\boldsymbol{Z}_p$-manifolds with compatible weak symplectic structure. Perhaps the most interesting aspect of the computation is the difference between the cases p odd and p equals 2, reflecting the presence of 2-torsion in Ω_*^{Sp}.

I am grateful to Elmer Rees for discussing this problem with me and for showing me his version of the computations in [2] for real projective space.

Notation. Let KU( ), KO( ) and KSp( ) be the cohomology functors defined by unitary, orthogonal and symplectic bundles respectively. There are natural operators

$$\mathrm{KO}(X) \underset{p}{\overset{i}{\rightleftharpoons}} \mathrm{KU}(X) \underset{q}{\overset{j}{\rightleftharpoons}} \mathrm{KSp}(X)$$

and complex conjugation $c\colon \mathrm{KU}(X) \to \mathrm{KU}(X)$. These satisfy the relations

$$\begin{array}{lll} pi = 2 & qj = 1 + c & pc = p \\ ip = 1 + c & jq = 2 & jc = j. \end{array}$$

This notation will be used below without further comment.

The relation between K-theory and bordism. As in the SO- and U-bordism theories there is a natural isomorphism between $\Omega_n^{\mathrm{Sp}}(\boldsymbol{Z}_p)$ and $\Omega_n^{\mathrm{Sp}}(K(\boldsymbol{Z}_p, 1))$, defined by classifying a principal $\boldsymbol{Z}_p$-manifold as a principal $\boldsymbol{Z}_p$-bundle. Let T be the weakly symplectic $\boldsymbol{Z}_p$-action which defines the lens space

$$L^{4n-1}(p;\underbrace{1, \cdots, 1}_{2n \text{ times}}) = L_p^{4n-1}.$$

Received by the editors October 21, 1968.

When p equals 2, T reduces to the antipodal map A. Under the isomorphism above $[T, S^{4n-1}]$ corresponds to the inclusion of L_p^{4n-1} in the infinite join, which is the standard $K(\mathbf{Z}_p, 1)$. In [1] Conner and Floyd prove that the first cobordism Pontrjagin class $p_1 \colon \widetilde{\mathrm{K}}\mathrm{Sp}(X) \to \Omega^4_{\mathrm{Sp}}(X)$ is a monomorphism. If h_H is the canonical symplectic line bundle over $\mathbf{H}P(n)$, let $\nu \in \widetilde{\mathrm{K}}\mathrm{Sp}(\mathbf{H}P(n))$ denote the class of $(h_H - 1)$. The image of ν under p_1 is the cobordism class of the inclusion $u \colon \mathbf{H}P(n) \hookrightarrow \mathbf{H}P(\infty)$. Let $\pi \colon L_p^{4n+3} \to \mathbf{H}P(n)$ be the projection of the natural fibration, and consider the commutative diagram:

$$\begin{tikzcd}
\widetilde{\mathrm{K}}\mathrm{Sp}(L_p^{4n+3}) \arrow[r, "p_1"'] & \Omega^4_{\mathrm{Sp}}(L_p^{4n+3}) \arrow[r, "\sim", "D"'] & \Omega^{\mathrm{Sp}}_{4n-1}(L_p^{4n+3}) \\
\widetilde{\mathrm{K}}\mathrm{Sp}(\mathbf{H}P(n)) \arrow[u, "\pi^!"] \arrow[r, "p_1"'] & \Omega^4_{\mathrm{Sp}}(\mathbf{H}P(n)). \arrow[u, "\pi^*"] &
\end{tikzcd}$$

The isomorphism D on the right is Poincaré duality.

$$p_1 \pi^! \nu = \{u\pi \colon L_p^{4n+3} \to \mathbf{H}P(\infty)\}.$$

Regard $\mathbf{H}P(\infty)$ as a submanifold of codimension 4 in itself. Duality, which is a variant of the Thom representation theorem for bordism, then implies that the final image of $\pi^!\nu$ is the bordism class $\{L_p^{4n-1} \hookrightarrow L_p^{4n+3}\}$. Finally an application of the cellular approximation theorem shows that the map

$$V_* \colon \Omega^{\mathrm{Sp}}_{4n-1}(L_p^{4n+3}) \to \Omega^{\mathrm{Sp}}_{4n-1}(\mathbf{Z}_p)$$

is also a monomorphism. This proves

LEMMA. *The order of* $[T, S^{4n-1}]$ *in* $\Omega^{\mathrm{Sp}}_{4n-1}(\mathbf{Z}_p)$ (p *any prime*) *is equal to the order of* $\pi^!\nu$ *in* $\widetilde{\mathrm{K}}\mathrm{Sp}(L_p^{4n+3})$.

Orders of the bordism classes.

PROPOSITION 1. *For* p *odd* $[T, S^{4n-1}]$ *has additive order* p^{k+1} *in* $\Omega^{\mathrm{Sp}}_{4n-1}(\mathbf{Z}_p)$, *where* $k(p-1) \leqq 2n+1 < (k+1)(p-1)$.

PROOF. Let $\pi_0 \colon L_p^{4n+3} \to \mathbf{C}P(2n+1)$ be the natural fibration, and consider the composition

$$\underbrace{L_p^{4n+3} \xrightarrow[S^1]{\pi_0} \mathbf{C}P(2n+1) \xrightarrow[S^2]{} \mathbf{H}P(n)}_{\pi}.$$

If h_C is the canonical line bundle over $\mathbf{C}P(2n+1)$, denote the class of (h_C-1) in $\tilde{K}U(CP(2n+1))$ by μ, and the pullback of μ along π_0 by σ. Since the operator j commutes with continuous maps, and the pullback of ν is $j\mu$,

$$\pi^!\nu = j\pi^!{}_0\mu = j\sigma.$$

The same argument as that used by T. Kambe in [3], only with $\tilde{K}\mathrm{Sp}$ replacing $\tilde{K}O$, shows that $\tilde{K}\mathrm{Sp}(L_p^{4n+3})$ consists of p-torsion. (The top dimensional cell cannot introduce a factor Z_2, since $4n+3 \not\equiv 5 \bmod 8$.) Hence for the pair of operators (j, q), where jq is multiplication by 2, q is a monomorphism and j an epimorphism. Composing in the reverse order, $qj(\sigma)=(1+c)\sigma$. From the properties of the complex line bundle over $CP(2n+1)$, and the naturality of q with respect to $\pi_0^!$, it follows that

$$q(j\sigma) = \sigma^2 - \sigma^3 + \sigma^4 - \cdots.$$

T. Kambe (op. cit.) has computed the group $\tilde{K}U(L_p^{4n+3})$, and it follows from his calculations that $q(j\sigma)$, and hence $j\sigma$, have order p^{k+1}, where $k(p-1)\leqq 2n+1<(k+1)(p-1)$. Proposition 1 now follows from the Lemma.

The situation when p equals 2 is more complicated, since in the presence of 2-torsion it is not clear that j is an epimorphism. Also in this special case we replace the symbol T by A.

PROPOSITION 2. (i) *$[A, S^{8n+3}]$ has order 2^{4n+3} in $\Omega_{8n+3}^{\mathrm{Sp}}(Z_2)$, and* (ii) *$[A, S^{8n+7}]$ has order 2^{4n+4} in $\Omega_{8n+7}^{\mathrm{Sp}}(Z_2)$.*

PROOF. The projection π reduces to the fibration of projective spaces $RP(4n+3)\to HP(n)$ and h_H pulls back to the canonical line bundle h_R over $RP(4n+3)$ regarded as a symplectic bundle. Denote the class of (h_R-1) in $\tilde{K}O(RP(4n+3))$ by λ, when $\pi^!\nu=ji(\lambda)$. It is not obvious that $ji(\lambda)$ is always a generator of $\tilde{K}\mathrm{Sp}(RP(4n+3))$, but from Bott periodicity and the fact that $i\lambda$ generates $\tilde{K}U(RP(4n+3))$, it follows that it is enough to look at the cokernel of the forgetful map p. Consider the long exact sequence of groups associated with the pair (U, O) having quotient homotopically equivalent to ΩO.

$$(*)\qquad \begin{aligned} \cdots \to \tilde{K}O^3X \xrightarrow{\delta} \tilde{K}O^2X \to \tilde{K}U^4X \xrightarrow{p} \tilde{K}O^4X \\ \to \tilde{K}O^3X \xrightarrow{\delta} \tilde{K}U^5X \to \cdots. \end{aligned}$$

From the computations of M. Fujii we have the particular cases:

$$X = \boldsymbol{R}P(8n+3), \qquad \to \boldsymbol{Z} \to \boldsymbol{Z}_2 + \boldsymbol{Z}_2 \to \boldsymbol{Z}_{2^{4n+1}} \twoheadrightarrow \boldsymbol{Z}_{2^{4n}} \xrightarrow{0} \boldsymbol{Z} \to \boldsymbol{Z} \to$$

$$X = \boldsymbol{R}P(8n+7), \qquad \to \boldsymbol{Z} \to 0 \to \boldsymbol{Z}_{2^{4n+3}} \xrightarrow{\cong} \boldsymbol{Z}_{2^{4n+3}} \xrightarrow{0} \boldsymbol{Z} \to \boldsymbol{Z} \to .$$

In both cases the map p is an epimorphism. Since we are only interested in the magnitudes of the various groups, we can use periodicity to replace the sequence (*) with that associated with the pair (U, Sp), provided we shift the dimensions by 4. It follows that j is also an epimorphism, and that $ji(\lambda)$ generates the cyclic group $\tilde{K}\mathrm{Sp}(\boldsymbol{R}P(4n+3))$. Proposition 2 now follows from the Lemma.

Additional remarks. At least when p is odd, it is possible to show that the submodule of $\tilde{\Omega}_*^{\mathrm{Sp}}(\boldsymbol{Z}_p)$ generated by $\{[T, S^{4n-1}], n \geqq 1\}$ is proper. The spectral sequence for $\tilde{\Omega}_*^{\mathrm{Sp}}(\boldsymbol{Z}_p)$ has nonzero terms at the $E_{r,s}^2$ level only when $s \equiv 0 \bmod 4$, $r \equiv 1 \bmod 2$, and hence collapses. This is a consequence of the isomorphism of $\tilde{\Omega}_*^{\mathrm{Sp}} \otimes \boldsymbol{Z}[1/2]$ with $\Omega_*^{\mathrm{SO}} \otimes \boldsymbol{Z}[1/2]$ as $\boldsymbol{Z}[1/2]$ polynomial algebras, [4]. It follows that the edge homormorphism from $\Omega_*^{\mathrm{Sp}}(\boldsymbol{Z}_p)$ to $H_*(\boldsymbol{Z}_p, \boldsymbol{Z})$ is an epimorphism, and that our construction has failed to pick up a complete set of module generators.

As in the oriented, and in contrast to the unitary theory, the structure of $\Omega_*^{\mathrm{Sp}}(\boldsymbol{Z}_2)$ does not follow as a special case from that of $\tilde{\Omega}_*^{\mathrm{Sp}}(\boldsymbol{Z}_p)$. The general pattern of Proposition 1 would suggest that $[A, S^{8n+3}]$ and $[A, S^{8n+7}]$ have orders 2^{4n+3} and 2^{4n+5} respectively. In the second case the predicted order differs by a factor of 2 from the actual order; no doubt this is a consequence of the 2-torsion structure of Ω_*^{Sp} itself.

References

1. P. E. Conner and E. E. Floyd, *The relation of cobordism to* K-*theories*, Springer-Verlag, New York, 1966.

2. M. Fujii, KO-*groups of projective spaces*, Osaka J. Math. **4** (1967), 141–149.

3. T. Kambe, *The structure of* K-*rings of the lens space and their applications*, J. Math. Soc. Japan **18** (1966), 135–146.

4. S. P. Novikov, *Homotopy properties of Thom complexes*, Mat. Sb. (N.S.) **57** (**92**) (1962), 407–442.

University of Hull