ON CENTRALIZERS OF INVOLUTIONS

MARCEL HERZOG

1. Introduction. The main purpose of this paper is to establish sufficient conditions for a group of even order to contain a normal elementary Abelian 2-subgroup of order at most 4 (Theorem 1). As a consequence it is shown that PSL(2, 5) is the only simple group which contains an involution $x$ with the following property: the Sylow 2-subgroup of the centralizer $C$ of $x$ in $G$ is a noncyclic group of order 4 which is normal in $C$ (Theorem 3).

Several corollaries are derived from Theorem 1. In particular, a direct proof is given of the fact that PSL(2, 5) is the only group which has no normal 2-complement, no normal elementary Abelian 2-subgroups of order less than 8 and which contains an involution with an elementary Abelian centralizer of order 4 (Theorem 2).

If $G$ is a group, $x \in G$ and $T$ is a subset of $G$, $C_G(x)$, $C_G(x)$, $I(T)$, $o(T)$, $o(x)$, $\langle T \rangle$, $T^*$, $Z(G)$ and $K(G)$ denote respectively: the centralizer of $x$ in $G$, the conjugate class of $x$ in $G$, the set of involutions in $T$, the number of elements in $T$, the order of $x$, the group generated by $T$, $T - \{1\}$, the center of $G$ and the largest normal subgroup of $G$ of odd order. If $P$ is a $p$-group then $\Omega_1(P)$ is the subgroup of $P$ generated by elements of $P$ of order $p$.

From now on $G$ will be a group of even order, $x$ a fixed involution of $G$, $K = K(G)$, $C = C_G(x)$, $I = I(C_G(x))$, $C_l(x) = C_G(x)$, and $S$ a fixed Sylow 2-subgroup of $G$ containing $x$ such that $S_0 = S \cap C = $ Sylow 2-subgroup of $C$. We are ready to state the results.

**Theorem 1.** Suppose that there exists $y \in I - C_G(x)$ such that

\[(*) \quad C_G(u) \cap C_G(y) \subseteq C_G(y)\]

for all $u \in I$. Then $\langle C_G(y) \rangle$ is a proper elementary Abelian normal 2-subgroup of $G$.

If, in addition, $I \cap \langle C_G(y) \rangle = \{y\}$, then $o(\langle C_G(y) \rangle) \leq 4$.

**Corollary 1.** Suppose that the following conditions hold:

(a) $I = I(C_G(u))$ for all $u \in C_G(x) \cap I$;

(b) $I(C_G(y)) = I(C_G(z))$ for all $y, z \in I - C_G(x)$. Then one of the following statements holds:

(i) $G$ has one class of involutions and $\langle I \rangle$ is an elementary Abelian normal 2-subgroup of $C$.

Received by the editors April 19, 1968 and, in revised form, June 17, 1968.

170
(ii) $G$ has at least two classes of involutions and it contains a proper elementary Abelian normal 2-subgroup.

**Corollary 2.** Suppose that $o(I) \leq 3$. Then one of the following statements holds.

(i) $S_0 = S$, $x$ is the only involution in $S$ and $(x)K$ is a normal subgroup of $G$.

(ii) $S_0 = S$, $S$ contains exactly 3 involutions and $(x)K$ is a proper normal subgroup of $G$.

(iii) $S_0 = S$, $G$ has one conjugate class of involutions.

(iv) $G$ has at least 2 classes of involutions and it contains a normal elementary Abelian subgroup of order at most 4.

Corollary 2 immediately yields

**Corollary 3.** Suppose that $o(I)^3$ and $G$ is simple. Then $S = S_0$ and $G$ has only one conjugate class of involutions.

In case that $C$ is elementary Abelian of order 4 we get the following

**Theorem 2.** Suppose that $C = \{1, x, y, xy\}$ is elementary Abelian and $G$ has neither a normal 2-complement nor a normal elementary Abelian 2-subgroup of order less than 8. Then $G \cong PSL(2, 5)$.

The following corollary is an easy consequence of Theorem 2, the results of Suzuki in [6] and the results of Feit and Thompson in [2].

**Corollary 4.** Let $G$ be a finite noncyclic simple group containing an element $w$ such that $o(C_0(w)) \leq 4$. Then $G$ is isomorphic to one of the following groups: $PSL(2, 5)$, $PSL(2, 7)$, $A_6$ and $A_7$.

Our final theorem requires the deep results of Gorenstein and Walter [5] with respect to groups with a dihedral Sylow subgroup of order 4.

**Theorem 3.** Suppose that $S_0 = \{1, x, y, xy\}$ is elementary Abelian, $S_0$ is normal in $C$ and $G$ is simple. Then $G \cong PSL(2, 5)$.

The proof of Theorem 1 utilizes the following lemma, which is of independent interest.

**Lemma.** Let $U$ be a subgroup of the group $H$ and let $w$ be an involution of $H$ which normalizes $U$ leaving fixed exactly two elements of $U$, 1 and $y$. Let $V$ be a normal, $w$-invariant noncyclic elementary Abelian subgroup of $U$ containing $y$. Then $V$ is a Sylow 2-subgroup of $U$, $o(V) = 4$, and $U/V$ is Abelian.
2. Proof of the Lemma, Theorem 1 and Corollary 1. We begin with the proof of the Lemma. Obviously \( y \) is an involution. First assume that \( o(V) = 4, V = \{1, y, z, yz\} \); then \( z^w = yz \). Suppose that \( U/V \) is not an Abelian group of odd order. Then \( w \) fixes an element of \( (U/V)^t \), say \( uV \). Thus one of the following holds:

\[
\begin{align*}
w^w &= uy & \text{and} & \quad u &= u^{w^2} = u \\
&= uz & & \quad = uy \\
&= uyz & & \quad = uy.
\end{align*}
\]

Hence we must have \( u^w = uy \); but then \( (uz)^w = (uy)(yz) = uz \) a contradiction. Thus \( U/V \) is an Abelian group of odd order. If \( o(V) > 4 \), then \( w \) fixes an element of \( (V/(y))^t \), say \( z(y) \), and \( V_0 = \langle z, y \rangle \) is a normal, \( w \)-invariant, elementary Abelian subgroup of \( V \) containing \( y, o(V_0) = 4 \), and by the first part \( V = V_0 \), a contradiction. The proof of the Lemma is complete.

To prove Theorem 1, suppose first that \( Cl_\sigma(y) \nsubseteq C_\sigma(y) \) and let \( t \in Cl_\sigma(y) - C_\sigma(y) \). By a result of Brauer and Fowler [1, p. 572], there exists \( w \in I(G) \) such that \( w \in I(C_\sigma(x)) \cap C_\sigma(t) \subseteq I \). Hence by (*) \( t \in C_\sigma(w) \cap Cl_\sigma(y) \subseteq C_\sigma(y) \) a contradiction. It follows that \( Cl_\sigma(y) \subseteq C_\sigma(y) \) and \( \langle Cl_\sigma(y) \rangle = H \) is a normal subgroup of \( G \) contained in \( C_\sigma(y) \). If \( C_\sigma(y) = G \), then \( H = \langle y \rangle \neq G \) and the theorem follows. If \( C_\sigma(y) \neq G \), then \( H \) is a proper normal subgroup of \( G \) and obviously \( y \in \Omega_2(P) \triangleleft G \) where \( P \) is the Sylow 2-subgroup of \( Z(H) \). Hence \( Cl_\sigma(y) \subseteq \Omega_2(P) \) and \( H \) is elementary Abelian. Finally suppose that \( o(H) \geq 8 \) and \( I \cap H = \{y\} \). Then \( x \) leaves only \( y \) and 1 fixed in \( H \) and by the Lemma \( o(H) = 4 \), a contradiction. Thus \( o(H) \leq 4 \) and the proof of Theorem 1 is complete.

It remains to prove Corollary 1. If \( I \subseteq Cl(x) \), then each element of \( I \) belongs to the center of some Sylow 2-subgroup of \( G \) and therefore \( G \) has one class of involutions. By (a), \( \langle I \rangle \) is an elementary Abelian normal 2-subgroup of \( C \) and (i) holds. Suppose finally that \( I \nsubseteq Cl(x) \) and let \( y \in I - Cl(x) \). It follows from (b) that the elements of \( I - Cl(x) \) commute with each other. Thus for all \( u \in I \cap Cl(x) \),

\[
C_\sigma(u) \cap Cl_\sigma(y) = I \cap Cl_\sigma(y) \subseteq C_\sigma(y),
\]

and for all \( u \in I - Cl(x) \),

\[
C_\sigma(u) \cap Cl_\sigma(y) = (C_\sigma(y)) \cap Cl_\sigma(y) \subseteq C_\sigma(y).
\]

It follows then by Theorem 1 that \( G \) has a proper normal elementary Abelian 2-subgroup.
3. Proof of Theorem 2 and Corollaries 2 and 4. We begin with Corollary 2. If \( o(I) = 1 \), then \( S_0 = S \), \( x \) is the only involution in \( S \) and by [3], \( (x)K \) is a normal subgroup of \( G \), as described in (i). If \( o(I) \neq 2 \), let \( o(I) = 3 \), \( I = \{ x, y, xy \} \). If no element of \( I \) is conjugate to \( x \) in \( G \), then \( N_S(S_0) = S_0 \), \( S = S_0 \), and by [3] \( (x)K \not\subset G \). Since \( o(I) = 3 \), \( (x)K \not\subset G \) and (ii) holds. If all the elements of \( I \) are conjugate in \( G \), then again \( S_0 = S \) and (iii) holds. Suppose finally that \( x \) is conjugate to \( xy \) in \( G \), but not to \( y \). Then \( I(C_o(xy)) = I \) and by Corollary 1, \( (C_l(y)) \) is a normal elementary Abelian 2-subgroup of \( G \). Hence, as either \( (C_l(y)) = \langle y \rangle \) or \( C_l(y) \) contains an element which does not commute with \( x \), \( I \cap (C_l(y)) = \{ y \} \) and by Theorem 1, \( o((C_l(y))) \leq 4 \), so that (iv) holds. This completes the proof of Corollary 2.

We continue with Theorem 2. If \( C = S \), then by Lemma 15.2.4 of [4], \( G \) has only one class of involutions and \( N = N_o(C) \cong PSL(2, 3) \). Thus \( C \) contains the centralizer of each of its nonunit elements and by Theorem 9.3.2 in [4], due to Suzuki, \( G \) is a Zassenhaus group of degree 5 with \( N \) the subgroup fixing a letter. Thus \( N \) is a Frobenius group with complement of order \( e = 3 \) and kernel of order \( n = 4 \). Since \( e \) is odd and \( e = n - 1 \), it follows from Theorems 13.3.5 and 13.1.1 in [4], due to Zassenhaus, that \( G \cong PSL(2, 4) \cong PSL(2, 5) \). Next assume that \( C \not\subset S \) and let \( y \in C \cap Z(S) \). As \( N_S(C) \not\subset C \), \( xy \) is conjugate to \( x \) in \( G \) and \( C_o(xy) = C \). Since \( y \) is not conjugate to \( x \) in \( G \), it follows from Theorem 1 that \( (C_l(y)) \) is a normal elementary Abelian 2-subgroup of \( G \). As before \( I \cap (C_l(y)) = \{ y \} \), and it follows by Theorem 1 that \( o((C_l(y))) \leq 4 \) in contradiction to our assumptions. The proof is complete.

It remains to prove Corollary 4. If \( o(C_o(w)) = 2 \), then \( G \) is not simple. If \( o(C_o(w)) = 3 \), then by [2], \( G \) is isomorphic either to \( PSL(2, 5) \) or to \( PSL(2, 7) \). If \( o(C_o(w)) = 4 \) and \( o(w) = 4 \), then by [6], \( G \) is isomorphic to one of the groups \( PSL(2, 7) \), \( A_6 \) and \( A_7 \). If, finally, \( o(C_o(w)) = 4 \) and \( o(w) = 2 \), then by Theorem 2, \( G \cong PSL(2, 5) \).

4. Proof of Theorem 3. If \( S = S_0 \), then by [5], \( G \cong PSL(2, q) \), \( q > 3 \). If \( q \) is even, then \( G \cong PSL(2, 4) \cong PSL(2, 5) \). If \( q \) is odd, then the centralizer \( C \) of an involution of \( G \) is a dihedral group of order \( q + e \), \( e = \pm 1 \). For \( S \) to be normal in \( C \), \( q + e = 4 \) and \( q = 5 \). Thus again \( G \cong PSL(2, 5) \). Suppose next that \( S_0 \not\subset S \), \( \{ y \} = Z(S) \cap S_0 \). Then \( N_S(S_0) \not\subset S_0 \), \( xy \) is conjugate to \( x \) in \( G \) and \( S_0 \) is the normal Sylow 2-subgroup of \( C_o(xy) \). As \( y \) is not conjugate to \( x \) in \( G \), it follows from Corollary 1 that \( G \) contains a proper, nontrivial, normal subgroup, in contradiction to the simplicity of \( G \). The proof is complete.
References

1. R. Brauer and K. A. Fowler, On groups of even order, Ann. of Math. 44 (1943), 57–79.


University of California, Santa Barbara