ON AN INEQUALITY OF LYAPUNOV

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Nehari [1] claims that if an interval $[a, b]$ contains n zeros of a nontrivial solution of

$$(1) \qquad y^{(n)} + p_n y^{(n-1)} + \cdots + p_1 y = 0$$

then

$$(2) \qquad \sum_{k=1}^{n} 2^k (b - a)^{n-k} \int_a^b |p_k| \, dt \geqq 2^{n+1}.$$

According to Fink and St. Mary [2] the proof of (2) given in [1] is incorrect, and therefore the inequality is, as yet, undecided. For $n = 2$ and $p_2 \equiv 0$, (2) is known to be correct and also the best possible. This case was first proved by Lyapunov and is generally referred to as Lyapunov's theorem. A recent proof of this result may be found in Hochstadt [3]. In [2] a similar technique is used to prove (2) for $n = 2$, where p_2 is merely integrable on $[a, b]$. In fact, one can extract from [2] the inequality

$$(3) \qquad \left[(b - a) \int_a^b |p_1| \, dt \right]^{1/2} + \frac{1}{2} \int_a^b |p_2| \, dt \geqq 2.$$

The purpose of this note is to provide a generalization of (3) to certain nth order differential equations.

THEOREM. *Consider the differential equation*

$$(4) \qquad y^{(n)} - p y^{(n-1)} - q y = 0, \qquad n \geqq 2,$$

where p and q are integrable on $[a, b]$. Suppose that a nontrivial solution of (4) has at least n zeros on $[a, b]$. Then

$$(5) \qquad \left[(b - a)^{n-1} \int_a^b |q| \, dt \right]^{1/n} + \frac{1}{n} \int_a^b |p| \, dt \geqq 2.$$

Clearly for $n = 2$, (5) reduces to (3). In order to prove (5) we reduce (4) to a system, by letting

$$x_i = y^{(i-1)}, \qquad i = 1, 2, \cdots, n,$$

so that

$$x_i' = x_{i+1}, \qquad i = 1, 2, \cdots, n - 1,$$

(6)

$$x_n' = px_n + qx_1.$$

Since y vanishes n times on $[a, b]$, each x_i vanishes at least once on that interval. We can, therefore, split $[a, b]$ into two subintervals $[a, c]$ and $[c, b]$ where $a < c < b$ such that on each, each x_i vanishes at least once.

First we shall consider the interval $[a, c]$, and let $\bar{x}_i$ denote the maximum of $|x_i|$ on that interval. Using (6) and the fact that each x_i vanishes at some point on $[a, c]$ we have

(7)
$$\bar{x}_i \leqq \bar{x}_{i+1}(c - a), \qquad i = 1, 2, \cdots, n - 1,$$

(8)
$$|x_n| \leqq \bar{x}_1 \int_a^c |q|\, dt + \int_t^c |p|\,|x_n|\, dt,$$

where $x_n(c) = 0$. From (7) we see that

$$\bar{x}_1 \leqq \bar{x}_n(c - a)^{n-1}$$

and combined with (8) we finally have

(9)
$$|x_n| \leqq \bar{x}_n(c - a)^{n-1} \int_a^c |q|\, dt + \int_t^c |p|\,|x_n|\, dt.$$

From (9) by means of Gronwall's inequality [4, p. 24] we find that

$$\bar{x}_n \leqq \bar{x}_n(c - a)^{n-1} \int_a^c |q|\, dt \exp \int_a^c |p|\, dt$$

and finally

(10)
$$\int_a^c |q|\, dt \geqq \frac{\exp \int_a^c |p|\, dt}{(c - a)^{n-1}}.$$

Similarly

(11)
$$\int_c^b |q|\, dt \geqq \frac{\exp \int_c^b |p|\, dt}{(b - c)^{n-1}}.$$

We can combine (10) and (11) and use the inequality

$$A^n/a^{n-1} + B^n/b^{n-1} \geqq (A + B)^n/(a + b)^{n-1}$$

to obtain

$$(12) \qquad (b-a)^{n-1} \int_a^b |q|\, dt \geqq \left[\exp\left(-\frac{1}{n}\int_a^c |p|\, dt\right) \right.$$
$$\left. + \exp\left(-\frac{1}{n}\int_c^b |p|\, dt\right) \right]^n.$$

(12) is an interesting inequality in its own right and for $n=2$ is also stated in [2]. In order to derive the simpler and desired inequality (5) we use the fact that

$$\exp(-x) \geqq 1 - x$$

in (12) and extract the nth root of both sides. Then

$$\left[(b-a)^{n-1} \int_a^b |q|\, dt \right]^{1/n} \geqq 2 - \frac{1}{n}\left(\int_a^c |p|\, dt + \int_c^b |p|\, dt \right)$$
$$= 2 - \frac{1}{n}\int_a^b |p|\, dt,$$

which is equivalent to (5).

BIBLIOGRAPHY

1. Z. Nehari, *On an inequality of Lyapunov*, Studies in Mathematical Analysis and Related Topics, Stanford Univ. Press, Stanford, Calif., 1962, pp. 256–261.
2. A. M. Fink and D. F. St. Mary, *On an inequality of Nehari*, Proc. Amer. Math. Soc. 21 (1969), 640–642.
3. H. Hochstadt, *A new proof of a stability estimate of Lyapunov*, Proc. Amer. Math. Soc. 14 (1963), 525–526.
4. P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964.

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