SHORTER NOTE

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

ORIENTABILITY OF HYPERSURFACES IN R^n

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We give an elementary proof for the fact that a hypersurface of codimension 1 in R^n is always orientable [a surface of the type we consider is, by definition, a closed, not necessarily compact, subset of R^n that near each of its points is the set of zeros of some real-valued C^∞-function with nonzero gradient]. Another way to say this: a nonorientable differentiable manifold (without boundary) of dimension $n-1$ cannot be C^∞-embedded as closed subset of R^n; here "closed" is needed as the Möbius strip (without boundary) in R^3 shows. This is usually proved (for compact manifolds, without any differentiability hypothesis) by invoking Alexander duality [2].

Suppose we had such a nonorientable hypersurface M in R^n. We take "nonorientable" to mean: there exists a loop (closed curve) in M such that the normal to M, when transported around the loop in a continuous fashion, comes back with the opposite direction. By considering a point on the normal a small distance off M, moving it around the loop and then connecting along the normal from one side of M to the other, we manufacture a closed C^∞-curve γ in R^n that meets M in exactly one point, and is transversal to M at this point [the tangent to γ is not tangent to M]. γ can be contracted to a point in R^n; this amounts to a C^∞-map f of the unit disk D^2 into R^n that on the boundary S^1 of D^2 yields γ. We now make f transversal to M, i.e., we bring it into "general position" relative to M, so that $f(D^2)$ is not tangent to M at any common point; this can be done without changing f on S^1, since f is already transversal there (see [1], [3], [4], [5]). [The general transversality theorem looks formidable, but is really quite simple, particularly in the case of codimension 1; one modifies f, locally, by adding suitable affine-linear functions, multiplied by cutoff functions; in the analogous piecewise linear case one shifts the images of the vertices of some triangulation of D^2 into general position.]

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By the main property of transversality the inverse image $f^{-1}(M)$ in D^2 is very well behaved: it consists of a disjoint union of simple closed C^∞-curves in the interior of D^2 and C^∞-arcs that meet S^1 exactly in their end points (NB, each arc ends in two distinct points). But $f^{-1}(M)$ has just one point on S^1; thus the whole thing is impossible—M cannot be nonorientable.

The same proof works in the piecewise linear case and with $\boldsymbol{R}^n$ replaced by any simply connected manifold. It generalizes to prove, e.g., the well-known fact that two closed submanifolds of $\boldsymbol{R}^n$ (one of them compact) of complementary dimensions cannot meet (transversally) in an odd number of points.

References

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2. P. Alexandroff and H. Hopf, *Topologie*, Springer, Berlin, 1935, p. 440.

3. H. J. Levine, *Singularities of differentiable mappings*, Mathematisches Institut der Universität Bonn, Bonn, 1959, p. 18.

4. J. Milnor, *Differential topology*, Lecture Notes, Princeton University, Princeton, N. J., 1959, Theorem 1.35, p. 22.

5. R. Thom, *Quelques propriētēs globales des variētēs differentiables*, Comment. Math. Helv. **28** (1954), 17–86, Theorem I.5.

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