ON THE KERNEL FUNCTION FOR THE
INTERSECTION OF TWO SIMPLY
CONNECTED DOMAINS

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1. Introduction. Let $D_1$ and $D_2$ be bounded simply connected domains in the complex plane each containing the origin and let $D$ be the component of $D_1 \cap D_2$ which contains the origin. It is clear that $D$ is simply connected. Let $\{W_n\}_{n=1}^\infty$ and $\{V_n\}_{n=1}^\infty$ be complete orthonormal sets in the spaces $L^2(D_1)$ and $L^2(D_2)$ respectively (if $G$ is a domain then $L^2(G)$ is the space of functions analytic in $G$ with $\int |f|^2 < \infty$). In this paper we show that the set $\{W_n: n = 1, 2, \ldots \} \cup \{V_n: n = 1, 2, \ldots \}$, with the domain restricted to $D$ in each case, spans $L^2(D)$. This means that given functions $f_1$ and $f_2$ which map $D_1$ and $D_2$ conformally onto the disk $|z| < 1$ one can construct a function $f$ which maps $D$ conformally onto the disk. This can be done as follows. Obtain $\{W_n\}_{n=1}^\infty$ and $\{V_n\}_{n=1}^\infty$ from $f_1$ and $f_2$ [3, p. 247] then construct a complete orthonormal set $\{Q_n\}_{n=1}^\infty$ for $L^2(D)$. The Bergman kernel function $K(z, \xi)$ for $D$ is

$$K(z, \xi) = \sum_{n=1}^\infty Q_n(z)Q_n(\xi).$$

We may then choose $f$ so that $f(0) = 0$ and $f'(z) = (\pi/K(0, 0))^{1/2} K(z, 0)$.

We observe that the result is clearly true in case the complements of $D_1$ and $D_2$ are closed domains. In this case the set $\{z^n: n = 0, 1, \ldots \}$ spans each of $L^2(D_1)$, $L^2(D_2)$ and $L^2(D)$, [2] and [3, p. 254].

2. Proof of the Theorem. Let $f_1$, $f_2$ and $f$ be functions which map $D_1$, $D_2$ and $D$ respectively onto the disk $|z| < 1$ with $f_1(0) = f_2(0) = f(0) = 0$. Suppose $g \in L^2(D)$. Since $\{(n + 1/\pi)^{1/2} b_n f^n(z) f'(z) \}_{n=0}^\infty$ is a complete orthonormal set in $L^2(D)$ [3, p. 247] we may write

$$g(z) = \sum_{n=0}^\infty (n + 1/\pi)^{1/2} b_n f^n(z) f'(z), \quad z \in D,$$

the series being absolutely and uniformly convergent on compact subsets of $D$ and

$$\int \int_D |g|^2 = \sum_{n=0}^\infty |b_n|^2 < \infty.$$

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Define

\[ g_\rho(z) = \sum_{n=0}^{\infty} (n + 1/\pi)^{1/2} \rho^n b_n(z) f'(z), \quad 0 < \rho < 1. \]

**Lemma 1.** There exists \( g^{(1)}_\rho(z) \) analytic in \( D_1 \) and \( g^{(2)}_\rho(z) \) analytic in \( D_2 \) such that

\[ g_\rho(z) = g^{(1)}_\rho(z) + g^{(2)}_\rho(z) \quad \text{when} \quad z \in D. \]

**Proof.** Let \( \{r_n\}_{n=1}^\infty \) be an increasing sequence of positive numbers such that \( \lim_{n \to \infty} r_n = 1 \). For \( 0 < r < 1 \), define

\[ D_1(r) = f_1^{-1}(rf_1(z)), \quad z \in D_1, \]
\[ D_2(r) = f_2^{-1}(rf_2(z)), \quad z \in D_2, \]
\[ D(r) = f^{-1}(rf(z)), \quad z \in D, \]

and let \( C^{(1)}_r \), \( C^{(2)}_r \) and \( C^{(0)} \) be the corresponding boundaries. Let \( D_r \) be the component of \( D^{(0)}_1 \cap D^{(0)}_2 \) which contains the origin and let \( \{R_n\}_{n=1}^\infty \) be an increasing sequence of positive numbers such that \( \lim_{n \to \infty} R_n = 1 \) and \( D^{(R_n)} \supset D_{r_n}, \quad n = 1, 2, \ldots \). We thus have a sequence \( \{D_{r_n}\} \) of domains such that \( D_{r_{n+1}} \supset D_{r_n}, \quad \bigcup_{n=1}^\infty D_{r_n} = D \) and a sequence \( \{C^{(R_n)}\}_{n=1}^\infty \) of closed curves contained in \( D \) such that \( D_{r_n} \) is contained in the interior of \( C^{(R_n)} \) and for \( z \in D_{r_n} \) and \( \xi \in C^{(R_n)} \),

\[ \min | \xi - z | = M_n > 0, \quad n = 1, 2, \ldots. \]

We now wish to find \( \{\Gamma^{(1)}_n\} \) and \( \{\Gamma^{(2)}_n\} \) such that \( \Gamma^{(1)}_n \cup \Gamma^{(2)}_n = C^{(R_n)} \) and such that \( \Gamma^{(1)}_n \cap D^{(R_n)}_1 \) and \( \Gamma^{(2)}_n \cap D^{(R_n)}_2 \) are empty, \( n = 1, 2, \ldots \). This is accomplished by setting \( \Gamma^{(1)}_n = C^{(R_n)} \cap D^{(R_n)}_1 \) and \( \Gamma^{(2)}_n = C^{(R_n)} \cap D^{(R_n)}_2 \).

For \( j = 1, 2, m \geq n = 1, 2, \ldots \) and \( z \in D^{(R_n)}_j \), define

\[ h^{(j)}_{m,n}(z) = \frac{1}{2\pi i} \int_{\Gamma^{(j)}_n} \frac{g_\rho(\xi)}{\xi - z} d\xi. \]

Then when \( z \in D_{r_n}, \ m \geq n \) we have

\[ g_\rho(z) = h^{(1)}_{m,n}(z) + h^{(2)}_{m,n}(z). \]

Further, if \( z \in D^{(R_n)}_1 \cap D^{(R_n)}_2 \) with \( m \geq n \) and \( m \geq n' \) then

\[ h^{(j)}_{m,n}(z) = h^{(j)}_{m,n'}(z). \]

We now fix \( n \) and show that \( \{h^{(j)}_{m,n}\}_{m=n}^\infty \) is a normal family in
Since the series (1) converges uniformly in $D^{(R_n)}$, we may write

$$h_{m,n}^{(j)}(z) = \sum_{k=0}^{\infty} ((k + 1)/\pi)^{1/2} \rho_{k} a_{m,n}^{(j)}(z; k), \quad z \in D^{(r_n)},$$

$j = 1, 2,$ where

$$a_{m,n}^{(j)}(z; k) = \frac{1}{2\pi i} \int_{\Gamma_{m}^{(j)}} \frac{f^{k}(\xi)f^{(j)}(\xi)}{\xi - z} \, d\xi.$$

But $w \in f(\Gamma_{m}^{(j)})$ implies $|w| = R_m < 1$ so

$$|a_{m,n}^{(j)}(z; k)| \leq \frac{1}{2\pi} \int_{f(\Gamma_{m}^{(j)})} \frac{|dw|}{M_n} \leq \frac{1}{2\pi} \int_{0}^{2\pi} \frac{d\theta}{M_n} = \frac{1}{M_n}.$$

Now (9) and (10) imply

$$|h_{m,n}^{(j)}(z)| \leq \sum_{k=0}^{\infty} ((k + 1)/\pi)^{1/2} \rho_{k} |b_{k}| \frac{1}{M_n}, \quad z \in D^{(r_n)}.$$

The series (11) converges by (2) and the fact that $\rho < 1$. Hence for fixed $n$ and $j = 1, 2$, the family $\{h_{m,n}^{(j)}\}_{m=n}^{\infty}$ is uniformly bounded in $D^{(r_n)}$ and is therefore a normal family. Let $I_{1}$ be a subset of the positive integers such that $\{h_{m,n}^{(1)}: m \in I_{1}\}$ converges to a function, say $h_{1}^{(1)}$, analytic in $D^{(r_n)}$. Let $I_{2}^{(2)} \subseteq I_{1}$ be such that $\{h_{m,n}^{(2)}: m \in I_{2}^{(2)}\}$ converges to $h_{2}^{(2)}$ analytic in $D^{(r_n)}$. Continuing in the same manner choose $I_{n}^{(n)} \subseteq I_{n}^{(n-1)}$ so that $\{h_{m,n}^{(n)}: m \in I_{n}^{(n)}\}$ converges to $h_{n}^{(n)}$ in $D^{(r_n)}$. Using (7) and (8), we then conclude

$$g_{\rho}(z) = h_{n}^{(1)}(z) + h_{n}^{(2)}(z) \quad \text{if} \quad z \in D_{n},$$

$$h_{n}^{(j)}(z) = h_{n}^{(j)}(z) \quad \text{if} \quad z \in D^{(r_n)} \cap D^{(r_n')}.$$
the series being absolutely and uniformly convergent on compact subsets of $D_j$, $j = 1, 2$.

**Proof.** The function $g_\phi(f_j^{-1}(w))$ is analytic in $|w| < 1$ so $g_\phi(f_j^{-1}(w)) = \sum_{n=0}^\infty a_n^j w^n$ and setting $z = f_j^{-1}(w)$, (13) follows.

Now let $\{W_n\}_{n=1}^\infty$ and $\{V_n\}_{n=1}^\infty$ be complete orthonormal sets in $L^2(D_1)$ and $L^2(D_2)$ respectively. Let $\{Q_n\}_{n=1}^\infty$ be an orthonormal set in $L^2(D)$ obtained by choosing a maximal linearly independent set from $\{W_n\} \cup \{V_n\}$ and orthonormalizing it.

Since $[f_j(z)], j = 1, 2$ is bounded, $f_j \in L^2(D_i), n = 0, 1, \ldots$, so we may write

$$f_1^n(z) = \sum_{k=1}^\infty a_{k,n} Q_k(z),$$

(14)

$$f_2^n(z) = \sum_{k=1}^\infty b_{k,n} Q_k(z), \quad z \in D,$$

the series converging uniformly and absolutely on compact subsets of $D$. Hence

$$g_\phi(z) = \sum_{n=0}^\infty \sum_{k=1}^\infty a_{k,n} Q_k(z) + \sum_{n=0}^\infty \sum_{k=1}^\infty b_{k,n} Q_k(z)$$

$$= \sum_{k=1}^\infty P_k Q_k(z), \quad z \in D.$$

(15)

The rearrangement is possible in (15) since the series (14) converge absolutely on compact subsets of $D$.

Now let $\epsilon > 0$ choose $\rho$ so that

$$\int_D \left| g - g_\phi \right|^2 = \sum_{n=0}^\infty (1 - \rho^n)^2 |b_n|^2 < \epsilon.$$

It is known [1, p. 2] that $\cdot \int_D \left| g - \sum_{n=1}^\infty c_n Q_n \right|^2$ is a minimum when

$$c_n = d_n = \int_D g(z) \overline{Q_n(z)} \, dx \, dy, \quad n = 1, 2, \ldots.$$

Hence we have

$$\int_D \left| g - \sum_{n=1}^\in\infty d_n Q_n \right|^2 \leq \int_D \left| g - \sum_{n=1}^\in\infty P_n Q_n \right|^2$$

$$= \int_D \left| g - g_\phi \right|^2 < \epsilon.$$
This implies \( g(z) = \sum_{n=1}^{\infty} d_n Q_n(z), \quad z \in D \) and that \( \sum_{n=1}^{\infty} |d_n|^2 = \iint_D |g(z)|^2 \, dx \, dy < \infty \). This proves the following theorem.

**Theorem.** The set \( \{ Q_n \}_{n=1}^{\infty} \) is complete in \( L^2(D) \).

### 3. Discussion

Two interesting questions remain open. Given \( g \in L^2(D) \) it would be desirable to obtain \( g^{(1)}(z) \) and \( g^{(2)}(z) \) analytic in \( D_1 \) and \( D_2 \) respectively so that

\[
g(z) = g^{(1)}(s) + g^{(2)}(z), \quad z \in D.
\]

However the present method does not yield this result. It seems necessary to use \( g_p(z) \) in order to obtain normal families. Then \( g_p(z) \) can be written in the form (4).

A second open question is whether we may require \( g^{(j)} \in L^2(D_j), j = 1, 2 \) in (4).

### References


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