EXTENDING HOMEOMORPHISMS OF $S^p \times S^q$

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The purpose of this note is to characterize those homeomorphisms of $S^p \times S^q$ onto itself which are concordant to the identity, and then to use this to classify, up to concordance, those which extend to homeomorphisms of $S^{p+q+1}$, regarded as $\partial (B^{p+1} \times B^{q+1}) = S^p \times B^{q+1} \cup B^{p+1} \times S^q$. We work throughout in the PL category, as defined and explicated in either [2] or [4].

Recall that homeomorphisms $f_0, f_1$ of $S^p \times S^q$ onto itself are said to be concordant (equivalently, weakly isotopic [3]) if there is a homeomorphism $F$ of $S^p \times S^q \times I$ onto itself such that $F(x, i) = (f_i(x), i)$ for $i = 0, 1$. $I = [0, 1]$ is the unit interval and we shall denote the identity function by 1. We will say that a homeomorphism of $S^p \times S^q$ onto itself is nice if it preserves orientation, extends to a homeomorphism of $S^{p+q+1}$, and induces the identity on $\pi_p(S^p \times S^q)$, where $1 \leq p \leq q$.

Theorem 1. A homeomorphism $h$ of $S^p \times S^q$ onto itself is nice if and only if it is concordant to the identity.

Proof. The case $p = q = 1$ follows from classical results, so we assume $q \geq p \geq 1$ and $q + p \geq 3$. Clearly homeomorphisms concordant to the identity are nice.

Suppose $h$ is a nice homeomorphism of $S^p \times S^q$, and let $h': S^{p+q+1} \to S^{p+q+1}$ be an extension of $h$; then $h'$ maps each complementary domain onto itself. For $p < q$ this is obvious, and for $p = q$ it follows from the assumption that $h$ induces the identity on $\pi_p(S^p \times S^q)$. Thus $h'$ must also preserve orientation. Restricting $h'$ to $S^p \times B^{q+1}$, it must induce the identity on $\pi_p(S^p \times B^{q+1}) = Z$; hence $h'| S^p \times 0$ is homotopic in $S^p \times B^{q+1}$ to the identity. Applying Zeeman's unknotting Theorem (Chapter 8 of [4], or Volume II of [2]), there is an isotopy $F$: $S^p \times B^{q+1} \times I \to S^p \times B^{q+1} \times I$ such that $F_0 = 1$, $F_1| S^p \times S^q \times I = 1$, and $F_t| S^p \times 0 \times I = 1$. Define $h'' = S^{p+q+1} \to S^{p+q+1}$ by $h'' = F_t h' \cup h'$; that is, $h''| S^p \times B^{q+1} = F_t h'$ and $h''| B^{p+1} \times S^q = h'$. Then $h''$ is an extension of $h$ which is isotopic to $h'$ by an isotopy preserving complementary domain, and such that $h''$ is the identity on the unknotted sphere $S^p \times 0$. According to Lemma 59, Chapter 8 of [4], there is an isotopy $G: S^{p+q+1} \times I \to S^{p+q+1} \times I$ from $h''$ to the identity such that $G| S^p \times 0 \times I = 1$. Now both $S^p \times B^{q+1} \times I$ and $G(S^p \times B^{q+1} \times I)$ are regular neighborhoods of $S^p \times 0 \times I$ in $S^{p+q+1} \times I$ meeting the boundary regul-

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larly in the same set; by the uniqueness of regular neighborhoods [1], there is a homeomorphism $H$ of $S^{p+q+1} \times I$ onto itself which is the identity on $S^{p+q+1} \times \partial I$ and such that $HG(S^p \times B^{q+1} \times I) = S^p \times B^{q+1} \times I$. Thus by restricting $HG$ to $S^p \times S^q \times I$ we obtain a concordance from $h$ to the identity, completing the proof of Theorem 1.

Notice that in the above proof $HGF$ is a concordance from $h'$ to the identity which preserves complementary domains. We exploit this to prove the following, which may be considered as a generalization of Theorem 3 of [3].

Corollary. If $h$ is a homeomorphism of $S^p \times B^{q+1}$ onto itself whose restriction to the boundary $S^p \times S^q$ is concordant to the identity, then $h$ is concordant to the identity.

We do not assume in this corollary that $p \leq q$. The proof is as follows: use a collar of $S^p \times S^q$ in $B^{p+1} \times S^q$ and the concordance from $h|_{S^p \times S^q}$ to the identity to extend $h$ to all of $S^{p+q+1}$. Then carry out the proof of Theorem 1; then, as remarked above, $HGF$ restricts to a concordance on $S^p \times B^{q+1}$ from $h$ to the identity.

The set of concordance classes of homeomorphisms of $S^p \times S^q$ onto itself forms a group under composition which we denote by $\text{Conc}(S^p \times S^q)$. Let $G_{p,q}$ be the subgroup consisting of concordance classes of homeomorphisms which extend to $S^{p+q+1}$. Assume $p \geq 1$.

Theorem 2. $G_{p,q}$ is isomorphic to $Z_2 \times Z_2$ if $p < q$, and $G_{p,p}$ is isomorphic to the dihedral group.

Proof. Let $f: S^p \to S^p$, $g: S^q \to S^q$ be orientation reversals, and denote their conical extensions by $C(f): B^{p+1} \to B^{p+1}$, $C(g): B^{q+1} \to B^{q+1}$. Define $\alpha, \beta: S^p \times S^q \to S^p \times S^q$ by $\alpha = f \times g$ and $\beta = f \times 1$. We may define extensions of $\alpha, \beta$ to $S^{p+q+1}$ by $\alpha' = f \times C(g) \cup C(f) \times g$ and $\beta' = f \times 1 \cup C(f) \times 1$. By examining their effect on orientation, or on $\pi_p(S^p \times S^q)$, one may see that no two of the homeomorphisms $1, \alpha, \beta, \alpha \beta = \beta \alpha$ are concordant. However, $\alpha \alpha$ and $\beta \beta$ are easily seen to be nice and hence concordant to the identity. This shows that the subgroup of $\text{Conc}(S^p \times S^q)$ generated by $\alpha$ and $\beta$ is isomorphic to $Z_2 \times Z_2$; but this subgroup is just $G_{p,q}$ if $p < q$, since one easily verifies that any extendable homeomorphism is concordant to either $1, \alpha, \beta, \text{ or } \alpha \beta$ by noting that its composition with one of them must be nice, and hence concordant to the identity.

If $p = q$, we add another generator $\lambda$ defined by $\lambda(x, y) = (y, x)$. Clearly $\lambda$ is extendable, $\lambda \lambda = 1$, and $\lambda$ commutes with $\alpha$; however, $\lambda \beta$ and $\beta \lambda$ are not concordant since they induce different isomorphisms.
of $\pi_p(S^p \times S^p)$. It is easy to check that $\alpha$, $\beta$, $\lambda$ generate a subgroup of $\text{Conc}(S^p \times S^p)$ of order 8; since there are only five groups of order 8, it is easy to determine this group: it is not commutative, and has too many elements of order two to be the Quaternion group, so it must be the dihedral group. But $\alpha$, $\beta$, $\lambda$ generate $G_{p,p}$; to see this, note that $1, \alpha, \beta, \alpha\beta, \lambda, \lambda\alpha, \lambda\beta, \beta\lambda$ are all distinct, because they induce different automorphisms of $\pi_p(S^p \times S^p)$, and any extendable homeomorphism is concordant to one of them because its composition with one of them must be nice.

References