A UNIQUENESS RESULT IN CONFORMAL MAPPING

JAMES A. JENKINS

1. Let Σ denote the family of functions $f(z)$ meromorphic and univalent in $|z|>1$ with Laurent expansion in the neighborhood of the point at infinity given by

$$f(z) = z + c_0 + c_1/z + \cdots + c_n/z^n + \cdots .$$

It is well known that, if f belongs to the family Σ and E denotes the complement of the image of $|z|>1$ under the mapping $w=f(z)$, the diameter of E belongs to the interval [2, 4]. The upper limit corresponds to the classical diameter theorem, obtained by elementary methods which automatically prove that equality is attained only for the slit mappings $f(z)=z+e^{i\alpha}z^{-1}+c$, α real, c constant. The proof that the diameter is at least 2 is also fairly elementary, indicated for example in [5, IV Abschn., No. 141]. However this proof does not in any evident manner provide a complete uniqueness statement. While the assertion appears in the literature, for example in the reference just given, that equality is attained only for the functions $f(z)=z+c$, c constant, the author has found no published proof of this statement. The object of this paper is to provide such a proof.

2. THEOREM. *If $f\in\Sigma$ and E, the complement of the image of $|z|>1$ under the mapping $w=f(z)$, has diameter 2, then $f(z)=z+c$, c constant.*

Evidently E cannot be a proper subset of any continuum of diameter 2; thus E must be convex and, in fact, a complete set of diameter 2 [2, p. 122]. It is thus a set of constant width 2 [2, Theorem 52]. The boundary of a bounded closed convex plane set is rectifiable [2, p. 88] and for a set of constant width 2 the length is 2π [2, p. 127].

If the appropriate branch of $(f'(z))^{1/2}$ has about the point at infinity the development

$$(f'(z))^{1/2} = 1 + \sum_{j=2}^{\infty} \beta_j z^{-j}$$

the above length is given by the formula $2\pi+2\pi\sum_{j=2}^{\infty}|\beta_j|^2$. The corresponding formula for functions in S [3] was given by Bieberbach [1]. The present formula can be found in [4]. (The latter paper may be somewhat difficult of access but in the present instance the formula is readily proved.) Thus we must have $\beta_j=0$, $j=2, \cdots$, and $f(z)=z+c$, c constant.

Received by the editors November 11, 1968.

Bibliography

1. L. Bieberbach, *Über die konforme Abbildung nahezu kreisförmiger Bereiche*, S.-B. Preuss. Akad. Wiss. Berlin, Phys.-Math. Kl. (1924), 181–188.

2. H. G. Eggleston, *Convexity*, Cambridge Univ. Press, Cambridge, 1958.

3. James A. Jenkins, *Univalent functions and conformal mapping*, Springer-Verlag, Berlin, 1958.

4. Y. Komatu, *A variational problem in conformal mappings of multiply-connected regions*, Rigaku **3** (1948), 3–6. (Japanese)

5. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*. Vol. II, Springer, Berlin, 1925.

Washington University