PROJECTIVE AND INJECTIVE OBJECTS IN THE CATEGORY OF BANACH SPACES

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Let F denote either the field of real numbers or the field of complex numbers. Let $\boldsymbol{B}\cdot$ be the category whose objects are Banach spaces over F and whose morphisms are continuous linear maps $f:A\rightarrow B$ with norm $|f| = \sup_{|x|\leq 1}|f(x)| \leqq 1$. The purpose of this note is to show that in $\boldsymbol{B}\cdot$, without the restrictions (restrictions that make certain monomorphisms isometric (into) mappings and epimorphisms onto mappings) made on monomorphisms and epimorphisms in [1], [3], and [5], the only injective object is the zero space and the only projective object is the zero space.

Notice that in $\boldsymbol{B}\cdot$ a morphism $f:A\rightarrow B$ is a monomorphism if and only if it is one-to-one and is an epimorphism if and only if $f[A] = \{b\in B \mid f(a)=b \text{ for some } a\in A\}$ is dense in B.

LEMMA 1. *F is a retract of every nonzero Banach space.*

PROOF. Let B be a nonzero Banach space in $\boldsymbol{B}\cdot$. Fix $b_0\in B$, $b_0\neq 0$. Let $b_0^*:B\rightarrow F$ be a morphism with $|b_0^*| = 1$ and $b_0^*(b_0) = |b_0|$ [**2**, Corollary 14, p. 65]. Then $b_0^*(b_0/|b_0|) = 1$. Define $f:F\rightarrow B$ by $f(m) = mb_0/|b_0|$. Then f is a morphism in $\boldsymbol{B}\cdot$ and $b_0^* \circ f = 1_F$ (the identity morphism of F). Therefore F is a retract of B.

LEMMA 2. *F is not injective in $\boldsymbol{B}\cdot$.*

PROOF. Let l_1 be the Banach space consisting of sequences $a = (a_i)_{i\in N}$, $a_i\in F$ and $N = \{1, 2, \cdots\}$, such that the norm $|a| = \sum_{i\in N}|a_i| < \infty$. Define $g:l_1\rightarrow l_1$ by $g((a_i)_{i\in N}) = (a_i/i+1)_{i\in N}$. Then g is a monomorphism in $\boldsymbol{B}\cdot$. Let $b = (1/i!)_{i\in N}$. Since $|b| = \sum_{i\in N} 1/i! = e-1$, $b\in l_1$. Now let $b^*:l_1\rightarrow F$ be a linear function with $|b^*| = 1$ and $b^*(b) = |b|$. There is no morphism $u:l_1\rightarrow F$ in $\boldsymbol{B}\cdot$ such that $u\circ g = b^*$, i.e. so that the following diagram commutes:

$$\begin{tikzcd} F & \\ l_1 \arrow[u, "b^*"] \arrow[r, "g"'] & l_1 \arrow[ul, dashed, "u"'] \end{tikzcd}$$

If such a morphism u existed in $\boldsymbol{B}\cdot$, then $(u\circ g)(b)$ must be $|b|$. However $g(b) = e-2 < 1$ but $|u(g(b))| = |b| > 1$. This is a contradiction.

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LEMMA 3. *F is not projective in $\boldsymbol{B}^{\cdot}$.*

PROOF. Let c_0 be the Banach space consisting of sequences $a=(a_i)_{i\in N}$, $a_i\in F$, and $N=\{1, 2, \cdots\}$, converging to zero with norm $|a|=\sup_{i\in N}|a_i|$. Define $g:c_0\to c_0$ by $g((a_i)_{i\in N})=(a_i/i)_{i\in N}$. Then g is an epimorphism and a monomorphism in $\boldsymbol{B}^{\cdot}$. Let b be the sequence $(0, 2, 0, 0, \cdots)$ in c_0. Then $g(b)=(0, 1, 0, 0, \cdots)$. Define $f: F\to c_0$ by $f(n)=(0, n, 0, 0, \cdots)$. There does not exist any morphism $u: F\to c_0$ in $\boldsymbol{B}^{\cdot}$ such that $g o u=f$, i.e. so that the following diagram commutes:

$$\begin{tikzcd} & F \arrow[dl, dashed, "u"'] \arrow[d, "f"] \\ c_0 \arrow[r, "g"] & c_0 \end{tikzcd}$$

If such a morphism u existed in $\boldsymbol{B}^{\cdot}$, then $u(1)$ must be b as g is one-to-one. However then $|u|=\sup_{|a|\leq 1}|u(a)|\geqq|u(1)|=2>1$. This is a contradiction.

PROPOSITION 1. *In $\boldsymbol{B}^{\cdot}$, if B is injective, then $B=0$.*

PROOF. If for objects A_1 and A_2 in a category C, A_1 is a retract of A_2 and A_2 is injective, then A_1 is injective. The result follows immediately from Lemmas 1 and 2.

PROPOSITION 2. *In $\boldsymbol{B}^{\cdot}$, if B is projective, then $B=0$.*

PROOF. If in a category C object A_1 is a retract of object A_2, and A_2 is projective, then A_1 is projective. The result follows from Lemmas 1 and 3.

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