

PROJECTIVE AND INJECTIVE OBJECTS IN THE CATEGORY OF BANACH SPACES

KENNETH POTHOVEN

Let F denote either the field of real numbers or the field of complex numbers. Let \mathcal{B} be the category whose objects are Banach spaces over F and whose morphisms are continuous linear maps $f: A \rightarrow B$ with norm $|f| = \sup_{|x| \leq 1} |f(x)| \leq 1$. The purpose of this note is to show that in \mathcal{B} , without the restrictions (restrictions that make certain monomorphisms isometric (into) mappings and epimorphisms onto mappings) made on monomorphisms and epimorphisms in [1], [3], and [5], the only injective object is the zero space and the only projective object is the zero space.

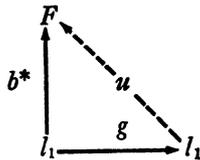
Notice that in \mathcal{B} a morphism $f: A \rightarrow B$ is a monomorphism if and only if it is one-to-one and is an epimorphism if and only if $f[A] = \{b \in B \mid f(a) = b \text{ for some } a \in A\}$ is dense in B .

LEMMA 1. *F is a retract of every nonzero Banach space.*

PROOF. Let B be a nonzero Banach space in \mathcal{B} . Fix $b_0 \in B$, $b_0 \neq 0$. Let $b_0^*: B \rightarrow F$ be a morphism with $|b_0^*| = 1$ and $b_0^*(b_0) = |b_0|$ [2, Corollary 14, p. 65]. Then $b_0^*(b_0/|b_0|) = 1$. Define $f: F \rightarrow B$ by $f(m) = mb_0/|b_0|$. Then f is a morphism in \mathcal{B} and $b_0^* \circ f = 1_F$ (the identity morphism of F). Therefore F is a retract of B .

LEMMA 2. *F is not injective in \mathcal{B} .*

PROOF. Let l_1 be the Banach space consisting of sequences $a = (a_i)_{i \in N}$, $a_i \in F$ and $N = \{1, 2, \dots\}$, such that the norm $|a| = \sum_{i \in N} |a_i| < \infty$. Define $g: l_1 \rightarrow l_1$ by $g((a_i)_{i \in N}) = (a_i/i+1)_{i \in N}$. Then g is a monomorphism in \mathcal{B} . Let $b = (1/i!)_{i \in N}$. Since $|b| = \sum_{i \in N} 1/i! = e-1$, $b \in l_1$. Now let $b^*: l_1 \rightarrow F$ be a linear function with $|b^*| = 1$ and $b^*(b) = |b|$. There is no morphism $u: l_1 \rightarrow F$ in \mathcal{B} such that $uog = b^*$, i.e. so that the following diagram commutes:

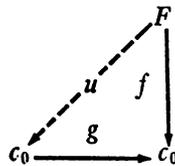


If such a morphism u existed in \mathcal{B} , then $(uog)(b)$ must be $|b|$. However $g(b) = e-2 < 1$ but $|u(g(b))| = |b| > 1$. This is a contradiction.

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LEMMA 3. *F is not projective in \mathcal{B} .*

PROOF. Let c_0 be the Banach space consisting of sequences $a = (a_i)_{i \in \mathbb{N}}$, $a_i \in F$, and $N = \{1, 2, \dots\}$, converging to zero with norm $|a| = \sup_{i \in \mathbb{N}} |a_i|$. Define $g: c_0 \rightarrow c_0$ by $g((a_i)_{i \in \mathbb{N}}) = (a_i/i)_{i \in \mathbb{N}}$. Then g is an epimorphism and a monomorphism in \mathcal{B} . Let b be the sequence $(0, 2, 0, 0, \dots)$ in c_0 . Then $g(b) = (0, 1, 0, 0, \dots)$. Define $f: F \rightarrow c_0$ by $f(n) = (0, n, 0, 0, \dots)$. There does not exist any morphism $u: F \rightarrow c_0$ in \mathcal{B} such that $gou = f$, i.e. so that the following diagram commutes:



If such a morphism u existed in \mathcal{B} , then $u(1)$ must be b as g is one-to-one. However then $|u| = \sup_{|a| \leq 1} |u(a)| \geq |u(1)| = 2 > 1$. This is a contradiction.

PROPOSITION 1. *In \mathcal{B} , if B is injective, then $B = 0$.*

PROOF. If for objects A_1 and A_2 in a category C , A_1 is a retract of A_2 and A_2 is injective, then A_1 is injective. The result follows immediately from Lemmas 1 and 2.

PROPOSITION 2. *In \mathcal{B} , if B is projective, then $B = 0$.*

PROOF. If in a category C object A_1 is a retract of object A_2 , and A_2 is projective, then A_1 is projective. The result follows from Lemmas 1 and 3.

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WESTERN MICHIGAN UNIVERSITY