A RELATION BETWEEN KILLING TENSOR FIELDS AND NEGATIVE PINCHED RIEMANNIAN MANIFOLDS

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1. Introduction. Let $M$ be a compact orientable Riemannian manifold. Let $M_P$ be the tangent space of the manifold $M$ at the point $P$. We denote by $(X, Y)$ and $\|X\|$ the scalar product of two vectors $X, Y \in M_P$ and the norm of the vector $X$, respectively, where the scalar product on the tangent space $M_P$ is induced by the Riemannian metric of the manifold $M$.

If $X, Y$ are two vectors of the tangent space $M_P$, then the curvature tensor field $R$ of the manifold $M$ and the two vectors $X, Y$ induce an endomorphism $R(X, Y)$ of $M_P$ into $M_P$. If $X, Y, Z, T$ are four vectors of $M_P$, then the Riemannian curvature tensor $R_4$ at $P$ can be considered as a quadrilinear mapping $R_4: M_P \times M_P \times M_P \times M_P \rightarrow \mathbb{R}$ which is defined by $R_4(X, Y, Z, T) \rightarrow \langle R(X, Y)Z, T \rangle$. Let $\lambda$ be a plane of the tangent space $M_P$ which is spanned by two linearly independent vectors $X, Y \in M_P$. The sectional curvature of the plane $\lambda$ is given by

$$\sigma(\lambda) = \sigma(X, Y) = -\frac{\langle R(X, Y)X, Y \rangle}{\|X\|^2 \|Y\|^2 - \langle X, Y \rangle^2}.$$  

We assume that the Riemannian manifold is compact orientable and negative $\delta$-pinched, that means its sectional curvature $\sigma(\lambda)$ satisfies the inequalities

$$-\Lambda \leq \sigma(\lambda) \leq -\Lambda \delta,$$

for every $\lambda \in M_P$ and $\forall P \in M$.

We can normalize the metric on the manifold $M$ such that the above inequalities become

$$-1 \leq \sigma(\lambda) \leq -\delta.$$

A Riemannian manifold $M$, whose sectional curvature satisfies the above inequalities, is called negative $\delta$-pinched.

Now, our results can be stated as follows: Let $M$ be a compact orientable negative $\delta$-pinched manifold. If the dimension of $M$ is even, $n = 2m$ (resp. odd, $n = 2m + 1$) and $\delta > 1/4$ (resp. $\delta > 2(m - 1)/(8m - 5)$), then there exists neither Killing tensor field of order 2 nor conformal Killing tensor field of order 2.

2. Let $M$ be a compact orientable negative $\delta$-pinched manifold.
We obtain a point $P$ of the manifold $M$ and consider a normal coordinate system on $M$ with origin the point $P$.

We also consider an orthonormal basis $\{X_1, \ldots, X_n\}$ of the tangent space $M_p$. If $\{X_i, X_j, X_k, X_l\}$ is a set of four vectors of the orthonormal basis $\{X_1, \ldots, X_n\}$, then the following formulas hold:

$$\langle R(X_i, X_j)X_k, X_l \rangle = R_{ijkl}, \quad \sigma(X_i, X_j) = \sigma_{ij} = - R_{ijij},$$

where $R_{ijkl}$ are the components of the Riemannian curvature.

If we apply the same technique as in [1, pp. 67–69], we obtain the following inequalities:

$$\sum_{i,j,k,l}|R_{ijkl}| \leq \frac{(1 - \delta)}{2}, \quad \sum_{i,j,k,l}|R_{ijkl}| \leq \frac{(1 - \delta)}{3}, \quad i \neq j \neq k \neq l,$$

because all the computations made in [1] for the sectional curvature ranging $(\frac{1}{4}, 1]$ if $n = 2m$, or $(\frac{2(m - 1)}{9m - 5}, 1]$ if $n = 2m + 1$ are valid without any change of the curvature ranging in any interval $[a, b]$.

3. Let $\xi = \{\xi(X_i, X_j) = \xi_{ij}\}$ be an exterior 2-form. This exterior 2-form is called a Killing 2-form if it satisfies the relation:

$$\nabla_X \xi(Y, Z) = - \nabla_Y \xi(X, Z), \quad \forall X, Y, Z \in T(M).$$

We consider the quadratic form $F(\xi)$ given by [2, p. 62]:

$$F(\xi) = \sum_{i,j} R_{ijij} \xi_{ij} + \frac{1}{2} \sum_{i,j,k,l} R_{ijkl} \xi_{ij} \xi_{kl}.$$

If the manifold is even dimension $n = 2m$ (resp. odd dimension $n = 2m + 1$) and $\delta > 1/4$ (resp. $\delta > 2(m - 1)/(8m - 5)$), then by means of the second of (2.1) and (2.2) with the same method as in [1, p. 70], we have $F(\xi) < 0$. It is well known [2, p. 67], if $F(\xi) < 0$, it implies $\xi = 0$.

Therefore, we can state the following theorem.

**Theorem I.** Let $M$ be a compact orientable negative $\delta$-pinched manifold. If the dimension of the manifold is even $n = 2m$ (resp. odd $n = 2m + 1$) and $\delta > 1/4$ (resp. $\delta > 2(m - 1)/(8m - 5)$), then there exists no Killing tensor field of order 2 on the manifold $M$.

4. Let $\mu = \{\mu(X_i, X_j) = \mu_{ij}\}$ be an exterior 2-form on the manifold $M$. This exterior 2-form is called a conformal Killing 2-form, if it satisfies the following conditions [2, p. 73]:

$$\nabla_X \mu_{ij} + \nabla_Y \mu_{ij} = 2\beta_{ij} \xi_{ij}, \quad \beta_{ij} = g^{ij} \nabla \mu_{ij}/\mu.$$

We can prove with the same technique as in §3. If the manifold $M$
is compact orientable and negative $\delta$-pinched, the dimension of the manifold is even $n = 2m$, (resp. odd $n = 2m + 1$), and $\delta > 1/4$ (resp. $\delta > 2(m - 1)/(8m - 5)$), then the quadratic form $F(n)$ is negative definite.

From the above and the known theorem [2, p. 73], we obtain the following theorem:

**Theorem II.** Let $M$ be a compact orientable negative $\delta$-pinched Riemannian manifold. If the dimension of the manifold is even $n = 2m$ (resp. odd $n = 2m + 1$) and $\delta > 1/4$ (resp. $\delta > 2(m - 1)/(8m - 5)$), then there exists no conformal Killing tensor field of order 2 on the manifold.

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**References**


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