TAME SUBSETS OF SPHERES IN $E^3$

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We present some conditions, in terms of special types of sequences of 3-manifolds with boundary, which are necessary and sufficient for a compact subset of a 2-sphere in $E^3$ to be tame. As a corollary to our results, we find that a tree-like subcontinuum $K$ of a 2-sphere in $E^3$ is tame if and only if $K$ can be described with trees of polyhedral 3-cells. Thus, while every tree-like continuum in $E^2$ can be described with trees of 2-cells in $E^2$ [1, Theorem 3], there exist tree-like continua in $E^3$ which are subsets of a 2-sphere and which cannot be described with trees of 3-cells in $E^3$.

A subset $K$ of a 2-sphere in $E^3$ is defined to be tame if there is a homeomorphism of $E^3$ onto itself that carries $K$ into a polyhedral sphere.

A sequence $\{M_i\}$ of sets is defined to be sequentially 1-ULC if for each $\epsilon > 0$ there exist an integer $k$ and a $\delta > 0$ such that, for $n > k$, each $\delta$-loop in $M_n$ can be shrunk to a point in an $\epsilon$-subset of $M_n$. (An $\epsilon$-set is a set with a diameter less than $\epsilon$.)

A set $F$ is defined to be $\epsilon$-dominated by a set $K$ if every point of $F$ is a subset of an $\epsilon$-arc which intersects $K$ and is a subset of $F$.

We say that a sequence $\{M_i\}$ of polyhedral 3-manifolds with boundary uniformly describes a compact set $K$ in $E^3$ if

1. for each $i$, $M_{i+1} \subset \text{Int } M_i$,
2. for each $i$, each component of $M_i$ is $1/i$-dominated by some component of $K$, and
3. $K = \bigcap_{i=1}^{\infty} M_i$.

A finite collection $T$ of polyhedral 3-cells is called a tree of cubes if the following conditions are satisfied:

1. Each two intersecting elements of $T$ have a 2-cell as their intersection.
2. The nerve of $T$ is a tree (i.e., a dendrite).

A tree of disks in $E^2$ can be defined similarly.

We say that a compact set $K$ in $E^3$ can be uniformly described with trees of cubes if for each $\epsilon > 0$ there exists a finite collection $C_1, \cdots, C_n$ of disjoint polyhedral cubes such that

1. $K \subset \bigcup_{i=1}^{n} \text{Int } C_i$,
2. for each $i$, there exists a tree $T_i$ of $\epsilon$-cubes whose union is

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Lemma 1. If $K$ is a continuum in $E^2$ that does not separate $E^2$, then for each $\varepsilon > 0$ there is a polyhedral disk $D$ in $E^2$ such that $K \subset \text{Int } D$ and $D$ is $\varepsilon$-dominated by $K$.

Proof. There exists a disk $D'$ in $E^2$ such that $K \subset \text{Int } D'$ and every point of $D'$ is within a distance $\varepsilon/2$ of $K$. Let $R_1, \ldots, R_n$ be triangular $\varepsilon/2$-disks in $\text{Int } D'$ such that $K \subset \bigcup_{i=1}^n \text{Int } R_i$, each $\text{Int } R_i$ intersects $K$, and $\text{Bd } R_1, \ldots, \text{Bd } R_n$ are in relative general position. Let $D$ denote $\bigcup_{i=1}^n R_i$ together with all of its bounded complementary domains in $E^2$. It follows that $D$ is a disk and that $K \subset \text{Int } D \subset \text{Int } D'$. Every point of $\bigcup_{i=1}^n R_i$ can be joined to $K$ with an $\varepsilon/2$-arc in $D$ and every point of $D - \bigcup_{i=1}^n R_i$ lies on a straight arc which intersects $\bigcup_{i=1}^n R_i$ and is of length less than $\varepsilon/2$. Thus the polyhedral disk $D$ is $\varepsilon$-dominated by $K$.

Lemma 2. If $K$ is a compact set in $E^3$ such that each component of $K$ is nondegenerate and $K$ can be uniformly described with a sequence $\{M_i\}$ of 3-manifolds with boundary, then for each $\varepsilon > 0$ there exist a compact subset $K'$ of $K$ and a sequence $\{M'_i\}$ of 3-manifolds with boundary such that

1. the diameters of the components of $K'$ have a positive lower bound,
2. each component of $K$ with a diameter no less than $\varepsilon$ is a subset of $K'$,
3. $\{M'_i\}$ uniformly describes $K'$, and
4. for each $i$, there is an integer $n_i$ so that each component of $M'_i$ is a component of $M_{n_i}$.

Proof. Let $H$ denote the union of all components of $K$ that have a diameter no less than $\varepsilon$. Let $n_1$ be a positive integer such that $1/n_1 < \varepsilon/4$, let $M'_i$ denote the union of all components of $M_{n_1}$ that intersect $H$, and let $H_1$ be the union of a finite number of components of $K$ such that each component of $M'_i$ is $1/n_1$-dominated by some component of $H_1$ and each component of $H_1$ $1/n_1$-dominates some component of $M'_i$. We proceed by induction to define sequences $\{n_i\}$, $\{M'_i\}$, and $\{H_i\}$. Suppose that $n_i$, $M'_i$, and $H_i$ have been defined for each $i < j$ and that each component of $H \cup (\bigcup_{i=1}^{j-1} H_i)$ has a diameter greater than $\varepsilon/2$. Let $n_j$ be a positive integer such that $n_j > n_{j-1}$ and each component of $H \cup (\bigcup_{i=1}^{j-1} H_i)$ has a diameter greater than $\varepsilon/2 + 2/n_j$. Let $M'_j$ denote the union of all components of $M_{n_j}$ that intersect $H \cup (\bigcup_{i=1}^{j-1} H_i)$, and let $H_j$ be the union of a finite number of components of $K$ such that each component of $M'_j$
is $1/n_j$-dominated by some component of $H_j$ and each component of $H_j$ $1/n_j$-dominates some component of $M'_j$. Thus we have defined the sequences $\{H_i\}$ and $\{M'_i\}$. Let $K' = H \cup (\text{cl } \bigcup_{j=1}^\infty H_j)$. The set $K'$ and the sequence $\{M'_i\}$ satisfy the requirements of the conclusion of Lemma 2.

**Theorem 1.** Suppose $K$ is a closed subset of a 2-sphere $S$ in $E^3$ such that $K$ does not separate $S$ and the components of $K$ are nondegenerate. Then in order that $K$ should be tame it is necessary and sufficient that there exist a sequence $\{M_i\}$ of 3-manifolds with boundary such that

1. $K$ is uniformly described by $\{M_i\}$,
2. each component of each $M_i$ is a polyhedral cube, and
3. $\{M_i\}$ is sequentially 1-ULC.

**Proof of Sufficiency.** It follows from Lemma 2 that $K$ is the union of a countable number of compact sets $K_1, K_2, \cdots$ such that, for each $i$, $K_i$ satisfies the sufficiency hypothesis of Theorem 1 and the diameters of the components of $K_i$ have a positive lower bound. That $K$ is tame will follow from [5, Theorem 1] and [6, Theorem 6], together with a proof that, for each $i$, $(*, K_i, S)$ is satisfied. (A definition of this property can be found in [6].) Thus to prove that $K$ is tame under the sufficiency hypothesis, we need only prove that $(*, K, S)$ is satisfied under the special assumption that the diameters of the components of $K$ have a positive lower bound. Making this assumption, we note that the proof of Theorem 1 of [3] shows that $(*, K, S)$ is satisfied if for each component $U$ of $E^3 - S$ the sequence $\{\text{cl}(U \cap \text{Bd } M_i)\}$ is sequentially 1-ULC in $E^3 - K$.

Let $U$ be a component of $E^3 - S$, and let $\epsilon$ be a positive number. Let $\delta$ be a positive number such that $\delta < \epsilon/4$ and each $\delta$-set on $S$ is a subset of an $\epsilon/4$-disk on $S$. Using the hypothesis that $\{M_i\}$ is sequentially 1-ULC, choose a positive integer $k$ and a positive number $\delta_i$ such that $1/k < \epsilon/4$ and, for each $i > k$, each $\delta_i$-loop in $M_i$ can be shrunk to a point in a $\delta$-subset of $M_i$. Choose $\eta > k$, and let $J$ be a simple closed curve in $\text{cl}(U \cap \text{Bd } M_\eta)$ of diameter less than $\delta_i$. Suppose that $J$ cannot be shrunk to a point in an $\epsilon$-subset of $E^3 - K$.

Let $C$ denote the component of $M_\eta$ that contains $J$. It follows from Dehn’s Lemma [7] that $J$ is the boundary of a $\delta$-disk $D$ so that $\text{Int } D \subseteq \text{Int } C$. By the above supposition and our choice of $\delta$, there exists an $\epsilon/4$-disk $E$ on $S$ such that $D \cap E \neq \emptyset$ and $D \cap S \subset E$. Let $D_1$ and $D_2$ be the two disks on $\text{Bd } C$ that are bounded by $J$. It follows from our supposition that each of $D_1$ and $D_2$ has a diameter no less than $\epsilon$. Thus $D_1$ and $D_2$ contain points $q_1$ and $q_2$, respectively, such that...
\[
\rho(q_1 \cup q_2, D \cup E) > 1/k.
\]

Let \( K_0 \) be a component of \( K \) such that \( K_0 \subset \text{Int} \, C \) and \( C \) is \( 1/n \)-dominated by \( K_0 \). There exists points \( p_1 \) and \( p_2 \) of \( K_0 \) which lie with \( q_1 \) and \( q_2 \), respectively, on \( 1/n \)-arcs \( A_1 \) and \( A_2 \) in \( C \). Since \( K_0 \subset \text{S}\cap \text{Int} \, C \), it follows that there exists an arc \( A \) in \( \text{S}\cap \text{Int} \, C \) with endpoints \( p_1 \) and \( p_2 \). The requirements in the choice of \( p_1 \) and \( p_2 \) imply that \( p_1 \cup p_2 \subset \text{S} - E \). Thus, since \( D \cap \text{S} \subset E \), it follows that there is an arc \( B \) from \( p_1 \) to \( p_2 \) such that \( B \cap D = \emptyset, B \subset \text{E}^3 - U \), and \( A \cup B \) is a simple closed curve. The way we constructed \( A \cup B \) relative to \( D \cup \text{Bd} \, C \) implies that \( A \cup B \) links \( J \). However, this is impossible as \( J \subset \text{S} \cup U \) and \( A \cup B \subset \text{E}^3 - U \). This contradiction enables us to conclude that \( J \) can be shrunk to a point in an \( \varepsilon \)-subset of \( \text{E}^3 - K \) and thus that \( \{ \text{cl}(U \cap \text{Bd} \, M_i) \} \) is sequentially \( 1 \)-ULC in \( \text{E}^3 - K \). As we indicated previously, this implies that \( (*, K, S) \) is satisfied and establishes the sufficiency of our condition.

**Proof of Necessity.** Using coordinates \((x, y, z)\) for \( \text{E}^3 \), we let \( P = \{(x, y, z) \mid z = 0\} \). We assume that \( K \subset P \) and that \( \text{Diam} \, K < 1/2 \). We will define the sequence \( \{M_i\} \) by induction.

Let \( M_1 \) denote a polyhedral cube of diameter less than 1 such that \( K \subset \text{Int} \, M_1 \). Suppose now that \( M_i \) has been defined for \( i < n \). It follows from Lemma 1 that there exist a finite sequence \( D_1, \ldots, D_m \) of polyhedral disks in \( P \) and a finite sequence \( K_1, \ldots, K_m \) of distinct components of \( K \) such that

\[
\begin{align*}
(4) & \quad \bigcup_{j=1}^{m} D_j \subset \text{Int} \, M_{n-1}, \\
(5) & \quad \text{each component of } K \text{ is a subset of some } \text{Int} \, D_j, \quad 1 \leq j \leq m, \\
(6) & \quad K_j \subset \text{Int} \, D_j, \quad 1 \leq j \leq m, \\
\end{align*}
\]

and

\[
\begin{align*}
(7) & \quad D_j \text{ is } 1/2n \text{-dominated by } K_j, \quad 1 \leq j \leq m.
\end{align*}
\]

There exists a finite sequence \( H_1, \ldots, H_r \) of disjoint disks such that

\[
\begin{align*}
(8) & \quad K \subset \bigcup_{j=1}^{r} \text{Int} \, H_j, \\
(9) & \quad K_j \subset \text{Int} \, H_j \subset H_j \subset \text{Int} \, D_j, \quad 1 \leq j \leq m, \\
\end{align*}
\]

and

\[
\begin{align*}
(10) & \quad \text{each } H_j, \quad 1 \leq j \leq r, \text{ is a subset of some } \text{Int} \, D_s, \quad 1 \leq s \leq m.
\end{align*}
\]
Now we identify disjoint closed sets $L_1, \ldots, L_m$ whose union is $K$ such that

(11) each $L_i$ is a finite union of sets of the form $K \cap H_j$,

and

(12) $K_j \subset L_j \subset \text{Int} D_j, \quad 1 \leq j \leq m.$

Let $\gamma$ be a positive number such that

(13) $\gamma < \frac{1}{3} \rho \left( \bigcup_{j=1}^m D_j, \text{Bd } M_{n-1} \right)$

and

(14) $\gamma < 1/10n.$

Let $z_1, \ldots, z_m$ be positive numbers such that

(15) $z_1 < z_2 < \cdots < z_m < \gamma.$

Let $P_j = \{(x, y, z) \mid z = s_j\}, 1 \leq j \leq m.$ There exists a piecewise-linear $\gamma$-homeomorphism $g$ of $E^3$ onto itself such that $g|L_j$ is a vertical projection of $L_j$ into $P_j, 1 \leq j \leq m.$ There exist a positive number $\sigma$ and disjoint polyhedral cubes $W_1, \ldots, W_m$ such that

(16) $\sigma < \gamma,$

(17) $W_j = \{(x, y, z) \mid (x, y, 0) \in D_j \text{ and } z_j - \sigma \leq z \leq z_j + \sigma\}.$

For each $j, 1 \leq j \leq m,$ let $C_j = g^{-1}(W_j)$ and let $M_n = \bigcup_{j=1}^m C_j.$ It follows from the way we have constructed $M_n$ that

(18) $M_n \subset \text{Int } M_{n-1},$

(19) $K \subset \text{Int } M_n,$

and

(20) each component of $M_n$ is $1/n$-dominated by some component of $K.$

In particular, $C_j$ is $1/n$-dominated by $K_j.$

With the above inductive procedure we have defined a sequence $\{M_i\}$ of 3-manifolds with boundary which satisfies requirements (1) and (2). It remains for us to show that $\{M_i\}$ is sequentially 1-ULC.

Let $\epsilon$ be a positive number, and let $k$ be an integer and $\delta$ a positive number such that $1/k < \epsilon$ and $\delta < \epsilon - 1/k.$ Let $L$ denote a $\delta$-loop in $M_i,$ where $i > k.$ We wish to show that $L$ can be shrunk to a point in an $\epsilon$-set in $M_i.$ There is a component $C$ of $M_i$ such that $L \subset C.$
Let $g_i$ denote the piecewise linear-$1/10i$-homeomorphism used in the inductive procedure to obtain $M_i$. It follows that $g_i(L)$ is a loop in $g_i(C)$ with a diameter less than $\delta + 1/5i$. From (14) and (17) we see that $g_i(L)$ is a subset of a 3-cell $V$ in $g_i(C)$ such that $\text{Diam } V < \delta + 2/5i$. It follows that $L \subset g_i^{-1}(V)$ and that

$$\text{Diam } g_i^{-1}(V) < \delta + 3/5i < \delta + 1/k < \varepsilon.$$ 

Thus we have shown that $\{M_i\}$ is sequentially 1-ULC.

**Theorem 2.** Suppose $K$ is a 1-dimensional compact subset of a 2-sphere $S$ in $\mathbb{E}^3$ such that $S - K$ is connected and the components of $K$ are nondegenerate. Then in order that $K$ should be tame it is necessary and sufficient that $K$ can be uniformly described with trees of cubes.

**Proof of Sufficiency.** From the hypothesis, it follows that there exists a sequence $\{M_i\}$ of polyhedral 3-manifolds with boundary that uniformly describes $K$ such that, for each $i$, each component of $M_i$ is a polyhedral cube that is the union of the elements of a tree of $1/i$-cubes. It readily follows that $\{M_i\}$ is sequentially 1-ULC. Thus it follows from Theorem 1 that $K$ is tame.

**Proof of Necessity.** We assume that $K$ is a subset of the plane $P$ as in the second part of Theorem 1. Now we can modify that proof by following a procedure described by Bing [1, Theorem 3] to require that each of the disks $D_1, \ldots, D_m$ be the union of the elements of a tree of $\varepsilon/2$-disks and that, for each such tree, some component of $K$ intersect each element of the tree. The process of moving the disks $D_1, \ldots, D_m$ into different planes and of thickening them to obtain cubes can be followed to show that $K$ can be uniformly described with trees of cubes.

**Corollary.** Suppose $K$ is a 1-dimensional subcontinuum of a 2-sphere $S$ in $\mathbb{E}^3$ such that $S - K$ is connected. Then $K$ is tame if and only if there exists a sequence $\{M_i\}$ of polyhedral cubes such that, for each $i$,

1. $K \subset M_{i+1} \subset \text{Int } M_i$, and
2. $M_i$ is the union of the elements of a tree of $1/i$-cubes.

**Remarks.** It is interesting to notice that there exists a 1-dimensional compact set $H$ in $\mathbb{E}^3$ such that $\mathbb{E}^3 - H$ is connected but $H$ cannot be uniformly described with trees of disks in $\mathbb{E}^3$. Such a set can be described as follows. Let

$$H = B \cup B_1 \cup B_2,$$
where
\[ B = \{ (x, y) \mid x = 0 \text{ and } -1 \leq y \leq 2 \}, \]
\[ B_1 = \{ (x, y) \mid 0 < x \leq 1, -1 \leq y \leq 3/4, \text{ and } y = \sin x^{-1} \}, \]
and
\[ B_2 = \{ (x, y) \mid 0 < x \leq 1, 1/4 \leq y \leq 2, \text{ and } y = 1 + \sin x^{-1} \}. \]

We observe that while each arc in \( E^3 \) can be described with a chain of open sets in \( E^3 \), it follows from the above corollary that no wild arc on a 2-sphere in \( E^3 \) can be described with a tree of cubes in \( E^3 \). However, there do exist wild arcs in \( E^3 \) which can be described with trees, or even chains, of cubes in \( E^3 \).

The next theorem gives a partial answer to the following question [2, pp. 78 and 82]: Is a 2-sphere \( S \) in \( E^3 \) tame if it is tame modulo a tame closed subset of \( S \) that has no point as a component? In a separate paper [4], Cannon will give an affirmative answer to the general form of the question.

**Theorem 3.** If the closed subset \( K \) of the 2-sphere \( S \) in \( E^3 \) is tame and has no point as a component, and \( S - K \) is connected and locally tame, then \( S \) is tame.

**Proof.** For each positive number \( \epsilon \), let \( K_\epsilon \) denote the union of all components of \( K \) that have a diameter no less than \( \epsilon \). It follows from Theorem 1 and its proof that \((\ast, K_\epsilon, S)\) is satisfied. Then [5, Theorem 1] implies that \((\ast, K, S)\) is satisfied, and [6, Theorem 15] implies therefore that \( S \) is tame.

**References**


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