TAME SUBSETS OF SPHERES IN E^3

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We present some conditions, in terms of special types of sequences of 3-manifolds with boundary, which are necessary and sufficient for a compact subset of a 2-sphere in E^3 to be tame. As a corollary to our results, we find that a tree-like subcontinuum K of a 2-sphere in E^3 is tame if and only if K can be described with trees of polyhedral 3-cells. Thus, while every tree-like continuum in E^2 can be described with trees of 2-cells in E^2 [1, Theorem 3], there exist tree-like continua in E^3 which are subsets of a 2-sphere and which cannot be described with trees of 3-cells in E^3.

A subset K of a 2-sphere in E^3 is defined to be *tame* if there is a homeomorphism of E^3 onto itself that carries K into a polyhedral sphere.

A sequence $\{M_i\}$ of sets is defined to be *sequentially* 1-ULC if for each $\epsilon>0$ there exist an integer k and a $\delta>0$ such that, for $n>k$, each δ-loop in M_n can be shrunk to a point in an ϵ-subset of M_n. (An ϵ-set is a set with a diameter less than ϵ.)

A set F is defined to be *ϵ-dominated* by a set K if every point of F is a subset of an ϵ-arc which intersects K and is a subset of F.

We say that a sequence $\{M_i\}$ of polyhedral 3-manifolds with boundary *uniformly describes* a compact set K in E^3 if

(1) for each i, $M_{i+1}\subset \operatorname{Int} M_i$,

(2) for each i, each component of M_i is $1/i$-dominated by some component of K, and

(3) $K=\bigcap_{i=1}^{\infty} M_i$.

A finite collection T of polyhedral 3-cells is called a *tree of cubes* if the following conditions are satisfied:

(1) Each two intersecting elements of T have a 2-cell as their intersection.

(2) The nerve of T is a tree (i.e., a dendrite).

A *tree of disks* in E^2 can be defined similarly.

We say that a compact set K in E^3 can be *uniformly described with trees of cubes* if for each $\epsilon>0$ there exists a finite collection $C_1, \cdots, C_n$ of disjoint polyhedral cubes such that

(1) $K\subset\bigcup_{i=1}^{n} \operatorname{Int} C_i$,

(2) for each i, there exists a tree T_i of ϵ-cubes whose union is

Received by the editors October 7, 1968.

[1] Work supported by NSF Grant GP-7058.

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C_i, and

(3) for each i, some component of K intersects each element of T_i.

LEMMA 1. *If K is a continuum in E^2 that does not separate E^2, then for each $\epsilon>0$ there is a polyhedral disk D in E^2 such that $K\subset \operatorname{Int} D$ and D is ϵ-dominated by K.*

PROOF. There exists a disk D' in E^2 such that $K\subset \operatorname{Int} D'$ and every point of D' is within a distance $\epsilon/2$ of K. Let $R_1, \cdots, R_n$ be triangular $\epsilon/2$-disks in $\operatorname{Int} D'$ such that $K\subset \bigcup_{i=1}^n \operatorname{Int} R_i$, each $\operatorname{Int} R_i$ intersects K, and $\operatorname{Bd} R_1, \cdots, \operatorname{Bd} R_n$ are in relative general position. Let D denote $\bigcup_{i=1}^n R_i$ together with all of its bounded complementary domains in E^2. It follows that D is a disk and that $K\subset \operatorname{Int} D\subset D \subset \operatorname{Int} D'$. Every point of $\bigcup_{i=1}^n R_i$ can be joined to K with an $\epsilon/2$-arc in D and every point of $D-\bigcup_{i=1}^n R_i$ lies on a straight arc which intersects $\bigcup_{i=1}^n R_i$ and is of length less than $\epsilon/2$. Thus the polyhedral disk D is ϵ-dominated by K.

LEMMA 2. *If K is a compact set in E^3 such that each component of K is nondegenerate and K can be uniformly described with a sequence $\{M_i\}$ of 3-manifolds with boundary, then for each $\epsilon>0$ there exist a compact subset K' of K and a sequence $\{M_i'\}$ of 3-manifolds with boundary such that*

(1) *the diameters of the components of K' have a positive lower bound,*

(2) *each component of K with a diameter no less than ϵ is a subset of K',*

(3) *$\{M_i'\}$ uniformly describes K', and*

(4) *for each i, there is an integer n_i so that each component of M_i' is a component of M_{n_i}.*

PROOF. Let H denote the union of all components of K that have a diameter no less than ϵ. Let n_1 be a positive integer such that $1/n_1<\epsilon/4$, let M_1' denote the union of all components of M_{n_1} that intersect H, and let H_1 be the union of a finite number of components of K such that each component of M_1' is $1/n_1$-dominated by some component of H_1 and each component of H_1 $1/n_1$-dominates some component of M_1'. We proceed by induction to define sequences $\{n_i\}$, $\{M_i'\}$, and $\{H_i\}$. Suppose that n_i, M_i', and H_i have been defined for each $i<j$ and that each component of $H\cup(\bigcup_{i=1}^{j-1} H_i)$ has a diameter greater than $\epsilon/2$. Let n_j be a positive integer such that $n_j>n_{j-1}$ and each component of $H\cup(\bigcup_{i=1}^{j-1} H_i)$ has a diameter greater than $\epsilon/2+2/n_j$. Let M_j' denote the union of all components of M_{n_j} that intersect $H\cup(\bigcup_{i=1}^{j-1} H_i)$, and let H_j be the union of a finite number of components of K such that each component of M_j'

is $1/n_j$-dominated by some component of H_j and each component of H_j $1/n_j$-dominates some component of M_j'. Thus we have defined the sequences $\{H_i\}$ and $\{M_i'\}$. Let $K' = H \cup (\text{cl} \bigcup_{i=1}^{\infty} H_i)$. The set K' and the sequence $\{M_i'\}$ satisfy the requirements of the conclusion of Lemma 2.

THEOREM 1. *Suppose K is a closed subset of a 2-sphere S in E^3 such that K does not separate S and the components of K are nondegenerate. Then in order that K should be tame it is necessary and sufficient that there exist a sequence $\{M_i\}$ of 3-manifolds with boundary such that*

(1) *K is uniformly described by $\{M_i\}$,*

(2) *each component of each M_i is a polyhedral cube, and*

(3) *$\{M_i\}$ is sequentially 1-ULC.*

PROOF OF SUFFICIENCY. It follows from Lemma 2 that K is the union of a countable number of compact sets $K_1, K_2, \cdots$ such that, for each i, K_i satisfies the sufficiency hypothesis of Theorem 1 and the diameters of the components of K_i have a positive lower bound. That K is tame will follow from [5, Theorem 1] and [6, Theorem 6], together with a proof that, for each i, $(*, K_i, S)$ is satisfied. (A definition of this property can be found in [6].) Thus to prove that K is tame under the sufficiency hypothesis, we need only prove that $(*, K, S)$ is satisfied under the special assumption that the diameters of the components of K have a positive lower bound. Making this assumption, we note that the proof of Theorem 1 of [3] shows that $(*, K, S)$ is satisfied if for each component U of $E^3 - S$ the sequence $\{\text{cl}(U \cap \text{Bd } M_i)\}$ is sequentially 1-ULC in $E^3 - K$.

Let U be a component of $E^3 - S$, and let ϵ be a positive number. Let δ be a positive number such that $\delta < \epsilon/4$ and each δ-set on S is a subset of an $\epsilon/4$-disk on S. Using the hypothesis that $\{M_i\}$ is sequentially 1-ULC, choose a positive integer k and a positive number δ_1 such that $1/k < \epsilon/4$ and, for each $i > k$, each δ_1-loop in M_i can be shrunk to a point in a δ-subset of M_i. Choose $n > k$, and let J be a simple closed curve in $\text{cl}(U \cap \text{Bd } M_n)$ of diameter less than δ_1. Suppose that J cannot be shrunk to a point in an ϵ-subset of $E^3 - K$. Let C denote the component of M_n that contains J. It follows from Dehn's Lemma [7] that J is the boundary of a δ-disk D so that $\text{Int } D \subset \text{Int } C$. By the above supposition and our choice of δ, there exists an $\epsilon/4$-disk E on S such that $D \cap E \neq \emptyset$ and $D \cap S \subset E$. Let D_1 and D_2 be the two disks on Bd C that are bounded by J. It follows from our supposition that each of D_1 and D_2 has a diameter no less than ϵ. Thus D_1 and D_2 contain points q_1 and q_2, respectively, such that

$$\rho(q_1 \cup q_2, D \cup E) > 1/k.$$

Let K_0 be a component of K such that $K_0 \subset \text{Int } C$ and C is $1/n$-dominated by K_0. There exists points p_1 and p_2 of K_0 which lie with q_1 and q_2, respectively, on $1/n$-arcs A_1 and A_2 in C. Since $K_0 \subset S \cap \text{Int } C$, it follows that there exists an arc A in $S \cap \text{Int } C$ with endpoints p_1 and p_2. The requirements in the choice of p_1 and p_2 imply that $p_1 \cup p_2 \subset S - E$. Thus, since $D \cap S \subset E$, it follows that there is an arc B from p_1 to p_2 such that $B \cap D = \emptyset$, $B \subset E^3 - U$, and $A \cup B$ is a simple closed curve. The way we constructed $A \cup B$ relative to $D \cup \text{Bd } C$ implies that $A \cup B$ links J. However, this is impossible as $J \subset S \cup U$ and $A \cup B \subset E^3 - U$. This contradiction enables us to conclude that J can be shrunk to a point in an ϵ-subset of $E^3 - K$ and thus that $\{\text{cl}(U \cap \text{Bd } M_i)\}$ is sequentially 1-ULC in $E^3 - K$. As we indicated previously, this implies that (*, K, S) is satisfied and establishes the sufficiency of our condition.

Proof of Necessity. Using coordinates (x, y, z) for E^3, we let $P = \{(x, y, z) \mid z = 0\}$. We assume that $K \subset P$ and that $\text{Diam } K < 1/2$. We will define the sequence $\{M_i\}$ by induction.

Let M_1 denote a polyhedral cube of diameter less than 1 such that $K \subset \text{Int } M_1$. Suppose now that M_i has been defined for $i < n$. It follows from Lemma 1 that there exist a finite sequence $D_1, \cdots, D_m$ of polyhedral disks in P and a finite sequence $K_1, \cdots, K_m$ of distinct components of K such that

(4) $$\bigcup_{j=1}^{m} D_j \subset \text{Int } M_{n-1},$$

(5) each component of K is a subset of some $\text{Int } D_j$, $\quad 1 \leqq j \leqq m$,

(6) $$K_j \subset \text{Int } D_j, \qquad 1 \leqq j \leqq m,$$

and

(7) D_j is $1/2n$-dominated by K_j, $\quad 1 \leqq j \leqq m$.

There exists a finite sequence $H_1, \cdots, H_r$ of disjoint disks such that

(8) $$K \subset \bigcup_{j=1}^{r} \text{Int } H_j,$$

(9) $$K_j \subset \text{Int } H_j \subset H_j \subset \text{Int } D_j, \qquad 1 \leqq j \leqq m,$$

and

(10) each H_j, $\quad 1 \leqq j \leqq r$, is a subset of some $\text{Int } D_s$, $\quad 1 \leqq s \leqq m$.

Now we identify disjoint closed sets $L_1, \cdots, L_m$ whose union is K such that

$$\text{(11)} \qquad \text{each } L_i \text{ is a finite union of sets of the form } K \cap H_j,$$

and

$$\text{(12)} \qquad K_j \subset L_j \subset \operatorname{Int} D_j, \qquad 1 \leqq j \leqq m.$$

Let γ be a positive number such that

$$\text{(13)} \qquad \gamma < \frac{1}{3}\rho\left(\bigcup_{j=1}^{m} D_j, \operatorname{Bd} M_{n-1}\right)$$

and

$$\text{(14)} \qquad \gamma < 1/10n.$$

Let $z_1, \cdots, z_m$ be positive numbers such that

$$\text{(15)} \qquad z_1 < z_2 < \cdots < z_m < \gamma.$$

Let $P_j = \{(x, y, z) \mid z = z_j\}$, $1 \leqq j \leqq m$. There exists a piecewise-linear γ-homeomorphism g of E^3 onto itself such that $g \mid L_j$ is a vertical projection of L_j into P_j, $1 \leqq j \leqq m$. There exist a positive number σ and disjoint polyhedral cubes $W_1, \cdots, W_m$ such that

$$\text{(16)} \qquad \sigma < \gamma,$$

$$\text{(17)} \qquad W_j = \{(x, y, z) \mid (x, y, 0) \in D_j \text{ and } z_j - \sigma \leqq z \leqq z_j + \sigma\}.$$

For each j, $1 \leqq j \leqq m$, let $C_j = g^{-1}(W_j)$ and let $M_n = \bigcup_{j=1}^{m} C_j$. It follows from the way we have constructed M_n that

$$\text{(18)} \qquad M_n \subset \operatorname{Int} M_{n-1},$$

$$\text{(19)} \qquad K \subset \operatorname{Int} M_n,$$

and

(20) each component of M_n is $1/n$-dominated by some component of K. In particular, C_j is $1/n$-dominated by K_j.

With the above inductive procedure we have defined a sequence $\{M_i\}$ of 3-manifolds with boundary which satisfies requirements (1) and (2). It remains for us to show that $\{M_i\}$ is sequentially 1-ULC.

Let ϵ be a positive number, and let k be an integer and δ a positive number such that $1/k < \epsilon$ and $\delta < \epsilon - 1/k$. Let L denote a δ-loop in M_i, where $i > k$. We wish to show that L can be shrunk to a point in an ϵ-set in M_i. There is a component C of M_i such that $L \subset C$.

Let g_i denote the piecewise linear-$1/10i$-homeomorphism used in the inductive procedure to obtain M_i. It follows that $g_i(L)$ is a loop in $g_i(C)$ with a diameter less than $\delta+1/5i$. From (14) and (17) we see that $g_i(L)$ is a subset of a 3-cell V in $g_i(C)$ such that Diam $V<\delta+2/5i$. It follows that $L\subset g_i^{-1}(V)$ and that

$$\text{Diam } \overset{-1}{g_i}(V) < \delta + 3/5i < \delta + 1/k < \epsilon.$$

Thus we have shown that $\{M_i\}$ is sequentially 1-ULC.

THEOREM 2. *Suppose K is a* 1*-dimensional compact subset of a* 2*-sphere S in E^3 such that $S-K$ is connected and the components of K are nondegenerate. Then in order that K should be tame it is necessary and sufficient that K can be uniformly described with trees of cubes.*

PROOF OF SUFFICIENCY. From the hypothesis, it follows that there exists a sequence $\{M_i\}$ of polyhedral 3-manifolds with boundary that uniformly describes K such that, for each i, each component of M_i is a polyhedral cube that is the union of the elements of a tree of $1/i$-cubes. It readily follows that $\{M_i\}$ is sequentially 1-ULC. Thus it follows from Theorem 1 that K is tame.

PROOF OF NECESSITY. We assume that K is a subset of the plane P as in the second part of Theorem 1. Now we can modify that proof by following a procedure described by Bing [1, Theorem 3] to require that each of the disks $D_1, \cdots, D_m$ be the union of the elements of a tree of $\epsilon/2$-disks and that, for each such tree, some component of K intersect each element of the tree. The process of moving the disks $D_1, \cdots, D_m$ into different planes and of thickening them to obtain cubes can be followed to show that K can be uniformly described with trees of cubes.

COROLLARY. *Suppose K is a* 1*-dimensional subcontinuum of a* 2*-sphere S in E^3 such that $S-K$ is connected. Then K is tame if and only if there exists a sequence $\{M_i\}$ of polyhedral cubes such that, for each i,*

(1) $K\subset M_{i+1}\subset \text{Int } M_i$, *and*

(2) M_i *is the union of the elements of a tree of* $1/i$-cubes.

REMARKS. It is interesting to notice that there exists a 1-dimensional compact set H in E^2 such that E^2-H is connected but H cannot be uniformly described with trees of disks in E^2. Such a set can be described as follows. Let

$$H = B \cup B_1 \cup B_2,$$

where

$$B = \{(x, y) \mid x = 0 \text{ and } -1 \leqq y \leqq 2\},$$
$$B_1 = \{(x, y) \mid 0 < x \leqq 1, -1 \leqq y \leqq 3/4, \text{ and } y = \sin x^{-1}\},$$

and

$$B_2 = \{(x, y) \mid 0 < x \leqq 1, 1/4 \leqq y \leqq 2, \text{ and } y = 1 + \sin x^{-1}\}.$$

We observe that while each arc in E^3 can be described with a chain of open sets in E^3, it follows from the above corollary that no wild arc on a 2-sphere in E^3 can be described with a tree of cubes in E^3. However, there do exist wild arcs in E^3 which can be described with trees, or even chains, of cubes in E^3.

The next theorem gives a partial answer to the following question [**2**, pp. 78 and 82]: Is a 2-sphere S in E^3 tame if it is tame modulo a tame closed subset of S that has no point as a component? In a separate paper [**4**], Cannon will give an affirmative answer to the general form of the question.

THEOREM 3. *If the closed subset K of the 2-sphere S in E^3 is tame and has no point as a component, and $S-K$ is connected and locally tame, then S is tame.*

PROOF. For each positive number ϵ, let K_ϵ denote the union of all components of K that have a diameter no less than ϵ. It follows from Theorem 1 and its proof that $(*, K_\epsilon, S)$ is satisfied. Then [**5**, Theorem 1] implies that $(*, K, S)$ is satisfied, and [**6**, Theorem 15] implies therefore that S is tame.

REFERENCES

1. R. H. Bing, *Snake-like continua*, Duke Math. J. **18** (1951), 653–663.

2. C. E. Burgess, "Surfaces in E^3" in *Topology seminar, Wisconsin*, 1965, Ann. of Math. Studies, No. 60, Princeton Univ. Press, Princeton, N. J., 1966, pp. 73–80.

3. C. E. Burgess and L. D. Loveland, *Sequentially 1-ULC surfaces in E^3*, Proc. Amer. Math. Soc. **19** (1968), 653–659.

4. J. W. Cannon, *Characterization of taming sets on 2-spheres*, Trans. Amer. Math. Soc. (to appear).

5. ———, *Singular side approximations of 2-spheres*, Notices Amer. Math. Soc. **15** (1968), 556.

6. L. D. Loveland, *Tame subsets of spheres in E^3*, Pacific J. Math. **19** (1966), 489–517.

7. C. D. Papakyriakopoulos, *On Dehn's lemma and the asphericity of knots*, Ann. of Math. (2), **66** (1957), 1–26.

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