A KRULL-SCHMIDT THEOREM FOR INFINITE SUMS OF MODULES

R. B. WARFIELD, JR.

In this note we prove a strengthened form of the well-known Krull-Schmidt-Azumaya theorem [2], using results of Crawley and Jónsson [3]. We apply this result to obtain a number of results on direct sums of modules, including generalizations of Kaplansky's theorem [4] that a projective module over a logical ring is free.

1. The exchange property. A module M (over an associative ring R with 1) has the *exchange property* if for any module G, if

$$G = M' \oplus C = \oplus_{i\in I} D_i$$

with $M' \cong M$, then there are submodules $D_i' \subseteq D_i$ such that $G = M' \oplus (\oplus_{i\in I} D_i')$. Crawley and Jónsson proved ([3, Theorem 7.1] or [6, Theorem 7]) that if G is a direct sum of countably generated modules, each with the exchange property, then any two direct sum decompositions of G have isomorphic refinements. If G is also a direct sum of indecomposable modules, $G = \oplus_{i\in I} M_i$, then we can conclude that any direct sum decomposition of G refines into a decomposition isomorphic to this one, and, in particular, any summand of G is also isomorphic to a direct sum of indecomposable modules, each isomorphic to one of the M_i. In [3], this result is proved in the context of the theory of general algebraic systems, while in [6] a version in Abelian categories is proved. Our first result characterizes the indecomposable modules which have the exchange property.

PROPOSITION 1. *An indecomposable module has the exchange property if and only if its endomorphism ring is local.*

PROOF. An elementary argument shows that we need only consider the case where the index set I is finite (see the reduction in [3, Lemma 5.1]). In this case we may actually assume that I has only two elements. Suppose, then, that $\mathrm{End}(M)$ is a local ring and that $G = M \oplus C = D \oplus E$. We show there are submodules D', E' of D and E with $G = M \oplus D' \oplus E'$. Let the projections to D and E (restricted to M) be θ_1 and θ_2, let the projection to M be π, and let the natural injections of M, D and E into G be ϕ_M, ϕ_1, and ϕ_2. Then $1_M = \pi\phi_1\theta_1 + \pi\phi_2\theta_2$ and since $\mathrm{End}(M)$ is local, one of these, say $\pi\phi_1\theta_1$, must be an automorphism. Let σ be the inverse of this automorphism. The endo-

Received by the editors December 19, 1968.

morphism $\eta = \phi_1\theta_1\sigma\pi$ of G is a projection, whose image N can be identified with the image of θ_1. If $D' = D \cap \ker(\eta)$, then $G = N \oplus D' \oplus E$. In this decomposition, the projection onto N is $\eta'\pi_D$, where π_D is the original projection onto D and η' is the restriction of η to D. Computation shows that $\eta'\pi_D\phi_M$ gives an isomorphism of M onto N, and this shows that $G = M \oplus D' \oplus E$.

Conversely, let M be an indecomposable module and suppose that $\operatorname{End}(M)$ is not a local ring. We will show that M does not have the exchange property. Since the endomorphism ring is not local, there are two endomorphisms f and g which are not automorphisms, such that $1_M = f - g$. Let $A = M_1 \oplus M_2$, where $M_1 \cong M_2 \cong M$, with projections π_1, π_2. We imbed M in $M_1 \oplus M_2$ by the homomorphism (f, g): $M \to M_1 \oplus M_2$, and we call the image M'. We also can imbed M in A by the diagonal map $(1_M, 1_M)$, and we call the image of this map $d(M)$. A standard computation shows that $A = M' \oplus d(M)$. If the exchange property were to hold for M, we would have either $A = M_1 \oplus M'$ or $A = M_2 \oplus M'$, since M_1 and M_2 are indecomposable. In the first case, this would mean that $\pi_2(f, g)$ was an isomorphism. This is impossible since it would imply that g was an automorphism. The second alternative is similarly impossible, so M does not have the exchange property.

THEOREM 1. *If M is a direct sum of indecomposable modules, $M = \oplus_{i\in I} M_i$, where each M_i is countably generated and has local endomorphism ring, then any other direct sum decomposition of M refines to a decomposition isomorphic to this one, and (in particular) any summand of M is again a direct sum of modules, each isomorphic to one of the original summands M_i.*

This result is closely related to the well-known theorem of Azumaya [2] which says that if a module M is a direct sum of indecomposable modules, $M = \oplus_{i\in I} M_i$, where each M_i has local endomorphism ring, then any indecomposable summand of M is isomorphic to one of the M_i and any two decompositions of M into indecomposable summands are isomorphic. A similar result was obtained for sums of indecomposable algebras with the exchange property by Crawley and Jónsson, [**3**, Theorem 5.3]. Proposition 3 below is an example of a result which follows from Theorem 1 but not from the Azumaya theorem.

If we restrict the number of summands to be finite, then Proposition 1 and Theorem 1 generalize to any Abelian category, as does the Azumaya theorem. See [**5**] for another proof and an application, and [**1**] for a special case and another application. For infinite sums we need to assume that the category satisfies axiom A6–5, in which case Proposition 1 and the Azumaya theorem carry over immediately. Theorem 1 is also valid if a suitable substitute for the countability hypothesis is provided. We refer to [6, Theorem 7] for details. The following application of Proposition 1 is also valid in any Abelian category.

PROPOSITION 2 (CANCELLATION PROPERTY). *Let M be a module with local endomorphism ring and suppose A and B are modules such that $A \oplus M \cong B \oplus M$. Then $A \cong B$.*

PROOF. Suppose $G = A \oplus M = B \oplus M'$, with $M' \cong M$. Then applying the exchange property for M, either $G = M \oplus B$ (so $B \cong A$ trivially) or $G = M \oplus B' \oplus M'$, with $B' \subseteq B$. In this case $A \cong B' \oplus M'$ and $B \cong B' \oplus M$, and the result is established.

2. Applications.

PROPOSITION 3. *If M is a module over a local ring R and M is a summand of a module N where $N = \oplus_{i \in I} C_i$, and each C_i is of the form R/I_i, where I_i is a two-sided ideal, then M is also a direct sum of cyclic modules, each isomorphic to one of the C_i.*

This includes as a special case the theorem of Kaplansky [4] that a projective module over a local ring is free. It also shows that a summand of a direct sum of cyclic modules over a commutative local ring is again a direct sum of cyclic modules.

For an example in Abelian group theory, we consider torsion-free groups divisible by all primes except a given prime p, or equivalently, modules over the ring R_p of rational numbers which can be written as a fraction with denominator prime to p. If M is a torsion-free R_p-module with no divisible summand and if $M/pM \cong Z/pZ$, then $\hat{M}$ (the p-adic completion of M) is isomorphic to $\hat{R}_p$—the additive group of p-adic integers—so M can be regarded as a submodule of $\hat{R}_p$. This natural imbedding actually makes M a pure submodule, so that if $x \in \hat{R}_p$ and $px \in M$, then $x \in M$. Also, $E = \operatorname{End}(M)$ is thereby imbedded as a pure subring of $\hat{R}_p$. Assume that M has finite rank, that K is the quotient field of $\hat{R}_p$ and L is the Q-subspace spanned by E. Then L is a finite-dimensional integral domain and hence a field, so if $x \in E$ has an inverse in $\hat{R}_p$, its inverse lies in $\hat{R}_p \cap L = E$. Hence E is a discrete valuation ring (in particular a local ring) whose maximal ideal is generated by multiplication by p. Theorem 1 then implies the following:

PROPOSITION 4. *If M is a torsion-free Abelian group with no divisible summand, which is divisible by all primes except one, namely p, and if M is a direct sum of finite rank groups, $M = \oplus_{i \in I} M_i$, where $M_i/pM_i \cong Z/pZ$, then any summand of M is again a direct sum of subgroups isomorphic to the M_i and any direct sum decomposition of M refines into one isomorphic to the original one.*

Suppose that R is any associative ring with 1, M is an R-module satisfying both the ascending and descending chain conditions (a.c.c. and d.c.c.) and f is an endomorphism of M. Then $M = R \oplus H$, where R is the set of $x \in M$ such that $f^n(x) = 0$ for some n, and H is the set of $x \in M$ in the image of f^n for all n (Fitting's lemma). Further, f restricted to H is an automorphism of H. From this it follows that if M is indecomposable then $\mathrm{End}(M)$ is a local ring and its maximal ideal is the set of nilpotent endomorphisms. These facts can be extended as follows:

LEMMA. *Let M be an R-module and M_i a family of fully invariant submodules, and f an endomorphism of M.*

(a) *If the M_i satisfy both chain conditions and M is the union of the M_i, then $M = R \oplus H$ where R is the set of $x \in M$ such that $f^n(x) = 0$ for some n, and H is the set of $x \in M$ for which there is an i such that $x \in f^n(M_i)$ for all n.*

(b) *If the M/M_i satisfy both chain conditions and $M =$ proj lim M/M_i then $M = R \oplus H$ where R is the set of $x \in M$ such that for each i there is an n with $f^n(x) \in M_i$, and H is the set of $x \in M$ in the image of f^n for all n.*

In both cases f restricts to an automorphism of H. In particular, if in either of these cases M is indecomposable, then $\mathrm{End}(M)$ *is a local ring.*

PROPOSITION 5. *Let R be a ring which is either left Noetherian or commutative and M a left R-module which is a direct sum of submodules satisfying the d.c.c. By a further decomposition, these summands may be assumed indecomposable. Then any summand of M is also a direct sum of submodules satisfying the d.c.c. and any direct sum decomposition of M refines into a decomposition isomorphic to the original one.*

PROOF. It suffices to show that over such a ring an indecomposable module N with d.c.c. satisfies condition (a) of the lemma. Let N_1 be the socle of N and N_{n+1} the inverse image in N of the socle of N/N_n. We claim that N is the union of the submodules N_n, and to show this it suffices to remark that any cyclic R-module satisfying the d.c.c. also satisfies the a.c.c., so that for $x \in N$, $Rx \subseteq N_n$ for some n.

PROPOSITION 6. *Let R be a left-Noetherian ring and I a two-sided ideal generated by a finite number of elements in the center of R, such that $\bigcap_{n>0} I^n = 0$, R is complete in the I-adic topology, and R/I satisfies the d.c.c. Then if M is a left R-module which is a direct sum of finitely generated modules, any summand of M is again a direct sum of finitely generated modules. Further, any direct sum decomposition of M refines into a direct sum of indecomposable, finitely generated modules, and this decomposition is unique up to isomorphism.*

PROOF. Clearly if R/I satisfies the d.c.c., so does R/I^n since R is left-Noetherian. Hence if N is finitely generated, N/I^nN satisfies both chain conditions. Therefore it suffices to show that if N is a finitely generated R-module, then $\bigcap_{n>0} I^nN = 0$, and N is complete in the I-adic topology. This actually holds without the hypothesis that R/I satisfies the d.c.c. Clearly a finitely generated module L with $\bigcap_{n>0} I^nL = 0$ is complete in the I-adic topology. We represent N as a quotient, $N = F/K$, where F is free and finitely generated. It is enough to show that $\bigcap_{n>0} I^nN = 0$, and since $\bigcap_{n>0} I^nN$ is just the closure of 0 in the I-adic topology, this is equivalent to showing that K is a closed submodule in the topology of F. Clearly $\bigcap_{n>0} I^nK = 0$, so K is complete, and we need only show that the I-adic topology on K agrees with the topology induced from the I-adic topology on F. This is a standard argument in commutative algebra (Bourbaki, *Algèbre commutative*, Chapter III, Corollary 1, p. 61) where R is assumed to be commutative, and the fact that I is generated by elements of the center of R is enough to assure that the same arguments are valid in this case.

COROLLARY. *If P is a projective module over a ring satisfying the hypotheses of Proposition* 6, *then P is a direct sum of indecomposable left ideals generated by idempotents and any two such decompositions of P are isomorphic.*

For examples, we remark that an Artinian ring satisfies these hypotheses trivially, and so does the group ring of a finite group over the p-adic integers.

BIBLIOGRAPHY

1. M. Atiyah, *On the Krull-Schmidt theorem with application to sheaves*, Bull. Soc. Math. France **84** (1956), 307–317.

2. G. Azumaya, *Corrections and supplementaries to my paper concerning Krull-Remak-Schmidt's theorem*, Nagoya Math. J. 1 (1950), 117–124.

3. P. Crawley and B. Jónsson, *Refinements for infinite direct decompositions of algebraic systems*, Pacific J. Math. **14** (1964), 797–855.

4. I. Kaplansky, *Projective modules*, Ann. of Math. **68** (1958), 372–377.

5. E. Walker, *Quotient categories and quasi-isomorphisms of Abelian groups*, Proc. Colloq. Abelian Groups (Tihany), Akad. Kiadó, Budapest, 1964, pp. 147–162.

6. R. B. Warfield, Jr., *Decompositions of injective modules*, Pacific J. Math. (to appear).

UNIVERSITY OF WASHINGTON