A KRULL-SCHMIDT THEOREM FOR INFINITE SUMS
OF MODULES

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In this note we prove a strengthened form of the well-known Krull-Schmidt-Azumaya theorem [2], using results of Crawley and Jónsson [3]. We apply this result to obtain a number of results on direct sums of modules, including generalizations of Kaplansky's theorem [4] that a projective module over a logical ring is free.

1. The exchange property. A module \( M \) (over an associative ring \( R \) with 1) has the exchange property if for any module \( G \), if

\[
G = M' \oplus C = \bigoplus_{i \in I} D_i
\]

with \( M' \cong M \), then there are submodules \( D_i' \subseteq D_i \) such that \( G = M' \oplus (\bigoplus_{i \in I} D_i') \). Crawley and Jónsson proved ([3, Theorem 7.1] or [6, Theorem 7]) that if \( G \) is a direct sum of countably generated modules, each with the exchange property, then any two direct sum decompositions of \( G \) have isomorphic refinements. If \( G \) is also a direct sum of indecomposable modules, \( G = \bigoplus_{i \in I} M_i \), then we can conclude that any direct sum decomposition of \( G \) refines into a decomposition isomorphic to this one, and, in particular, any summand of \( G \) is also isomorphic to a direct sum of indecomposable modules, each isomorphic to one of the \( M_i \). In [3], this result is proved in the context of the theory of general algebraic systems, while in [6] a version in Abelian categories is proved. Our first result characterizes the indecomposable modules which have the exchange property.

**Proposition 1.** An indecomposable module has the exchange property if and only if its endomorphism ring is local.

**Proof.** An elementary argument shows that we need only consider the case where the index set \( I \) is finite (see the reduction in [3, Lemma 5.1]). In this case we may actually assume that \( I \) has only two elements. Suppose, then, that \( \text{End}(M) \) is a local ring and that \( G = M \oplus C = D \oplus E \). We show there are submodules \( D', E' \) of \( D \) and \( E \) with \( G = M \oplus D' \oplus E' \). Let the projections to \( D \) and \( E \) (restricted to \( M \)) be \( \theta_1 \) and \( \theta_2 \), let the projection to \( M \) be \( \pi \), and let the natural injections of \( M \), \( D \) and \( E \) into \( G \) be \( \phi_M \), \( \phi_1 \), and \( \phi_2 \). Then \( 1_M = \pi \phi_1 \theta_1 + \pi \phi_2 \theta_2 \) and since \( \text{End}(M) \) is local, one of these, say \( \pi \phi_1 \theta_1 \), must be an automorphism. Let \( \sigma \) be the inverse of this automorphism. The endo-

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A morphism \( \eta = \phi \beta \sigma \pi \) of \( G \) is a projection, whose image \( N \) can be identified with the image of \( \theta_1 \). If \( D' = D \cap \ker(\eta) \), then \( G = N \oplus D' \oplus E \).

In this decomposition, the projection onto \( N \) is \( \eta' \pi_D \), where \( \pi_D \) is the original projection onto \( D \) and \( \eta' \) is the restriction of \( \eta \) to \( D \). Computation shows that \( \eta' \pi_D \phi_M \) gives an isomorphism of \( M \) onto \( N \), and this shows that \( G = M \oplus D' \oplus E \).

Conversely, let \( M \) be an indecomposable module and suppose that \( \text{End}(M) \) is not a local ring. We will show that \( M \) does not have the exchange property. Since the endomorphism ring is not local, there are two endomorphisms \( f \) and \( g \) which are not automorphisms, such that \( 1_M = f - g \). Let \( A = M_1 \oplus M_2 \), where \( M_1 \cong M_2 \cong M \), with projections \( \pi_1, \pi_2 \). We imbed \( M \) in \( M_1 \oplus M_2 \) by the homomorphism \( (f, g) : M \to M_1 \oplus M_2 \), and we call the image \( M' \). We also can imbed \( M \) in \( A \) by the diagonal map \( (1_M, 1_M) \), and we call the image of this map \( d(M) \).

A standard computation shows that \( A = M' \oplus d(M) \). If the exchange property were to hold for \( M \), we would have either \( A = M_1 \oplus M' \) or \( A = M_2 \oplus M' \), since \( M_1 \) and \( M_2 \) are indecomposable. In the first case, this would mean that \( \pi_2(f, g) \) was an isomorphism. This is impossible since it would imply that \( g \) was an automorphism. The second alternative is similarly impossible, so \( M \) does not have the exchange property.

**Theorem 1.** If \( M \) is a direct sum of indecomposable modules, \( M = \oplus_{i \in I} M_i \), where each \( M_i \) is countably generated and has local endomorphism ring, then any other direct sum decomposition of \( M \) refines to a decomposition isomorphic to this one, and (in particular) any summand of \( M \) is again a direct sum of modules, each isomorphic to one of the original summands \( M_i \).

This result is closely related to the well-known theorem of Azumaya [2] which says that if a module \( M \) is a direct sum of indecomposable modules, \( M = \oplus_{i \in I} M_i \), where each \( M_i \) has local endomorphism ring, then any indecomposable summand of \( M \) is isomorphic to one of the \( M_i \) and any two decompositions of \( M \) into indecomposable summands are isomorphic. A similar result was obtained for sums of indecomposable algebras with the exchange property by Crawley and Jónsson, [3, Theorem 5.3]. Proposition 3 below is an example of a result which follows from Theorem 1 but not from the Azumaya theorem.

If we restrict the number of summands to be finite, then Proposition 1 and Theorem 1 generalize to any Abelian category, as does the Azumaya theorem. See [5] for another proof and an application, and [1] for a special case and another application. For infinite sums
we need to assume that the category satisfies axiom A6–5, in which case Proposition 1 and the Azumaya theorem carry over immediately. Theorem 1 is also valid if a suitable substitute for the countability hypothesis is provided. We refer to [6, Theorem 7] for details. The following application of Proposition 1 is also valid in any Abelian category.

**Proposition 2 (Cancellation Property).** Let \( M \) be a module with local endomorphism ring and suppose \( A \) and \( B \) are modules such that \( A \oplus M \cong B \oplus M \). Then \( A \cong B \).

**Proof.** Suppose \( G = A \oplus M = B \oplus M' \), with \( M' \cong M \). Then applying the exchange property for \( M \), either \( G = M \oplus B \) (so \( B \cong A \) trivially) or \( G = M \oplus B' \oplus M' \), with \( B' \subseteq B \). In this case \( A \cong B' \oplus M' \) and \( B \cong B' \oplus M \), and the result is established.

2. **Applications.**

**Proposition 3.** If \( M \) is a module over a local ring \( R \) and \( M \) is a summand of a module \( N \) where \( N = \bigoplus_{i \in I} C_i \), and each \( C_i \) is of the form \( R/I_i \), where \( I_i \) is a two-sided ideal, then \( M \) is also a direct sum of cyclic modules, each isomorphic to one of the \( C_i \).

This includes as a special case the theorem of Kaplansky [4] that a projective module over a local ring is free. It also shows that a summand of a direct sum of cyclic modules over a commutative local ring is again a direct sum of cyclic modules.

For an example in Abelian group theory, we consider torsion-free groups divisible by all primes except a given prime \( p \), or equivalently, modules over the ring \( R_p \) of rational numbers which can be written as a fraction with denominator prime to \( p \). If \( M \) is a torsion-free \( R_p \)-module with no divisible summand and if \( M/pM \cong Z/pZ \), then \( \hat{M} \) (the \( p \)-adic completion of \( M \)) is isomorphic to \( \hat{R}_p \)—the additive group of \( p \)-adic integers—so \( M \) can be regarded as a submodule of \( \hat{R}_p \). This natural imbedding actually makes \( M \) a pure submodule, so that if \( x \in \hat{R}_p \) and \( px \in M \), then \( x \in M \). Also, \( E = \text{End}(M) \) is thereby imbedded as a pure subring of \( \hat{R}_p \). Assume that \( M \) has finite rank, that \( K \) is the quotient field of \( \hat{R}_p \) and \( L \) is the \( Q \)-subspace spanned by \( E \). Then \( L \) is a finite-dimensional integral domain and hence a field, so if \( x \in E \) has an inverse in \( \hat{R}_p \), its inverse lies in \( \hat{R}_p \cap L = E \). Hence \( E \) is a discrete valuation ring (in particular a local ring) whose maximal ideal is generated by multiplication by \( p \). Theorem 1 then implies the following:
Proposition 4. If $M$ is a torsion-free Abelian group with no divisible summand, which is divisible by all primes except one, namely $p$, and if $M$ is a direct sum of finite rank groups, $M = \bigoplus_{i \in I} M_i$, where $M_i/pM_i \cong \mathbb{Z}/p\mathbb{Z}$, then any summand of $M$ is again a direct sum of subgroups isomorphic to the $M_i$ and any direct sum decomposition of $M$ refines into one isomorphic to the original one.

Suppose that $R$ is any associative ring with 1, $M$ is an $R$-module satisfying both the ascending and descending chain conditions (a.c.c. and d.c.c.) and $f$ is an endomorphism of $M$. Then $M = R \oplus H$, where $R$ is the set of $x \in M$ such that $f^n(x) = 0$ for some $n$, and $H$ is the set of $x \in M$ in the image of $f^n$ for all $n$ (Fitting's lemma). Further, if $M$ is indecomposable then $\text{End}(M)$ is a local ring and its maximal ideal is the set of nilpotent endomorphisms. These facts can be extended as follows:

Lemma. Let $M$ be an $R$-module and $M_i$ a family of fully invariant submodules, and $f$ an endomorphism of $M$.

(a) If the $M_i$ satisfy both chain conditions and $M$ is the union of the $M_i$, then $M = R \oplus H$ where $R$ is the set of $x \in M$ such that $f^n(x) = 0$ for some $n$, and $H$ is the set of $x \in M$ for which there is an $i$ such that $x \in f^n(M_i)$ for all $n$.

(b) If the $M/M_i$ satisfy both chain conditions and $M = \text{proj lim } M/M_i$, then $M = R \oplus H$ where $R$ is the set of $x \in M$ such that for each $i$ there is an $n$ with $f^n(x) \in M_i$, and $H$ is the set of $x \in M$ in the image of $f^n$ for all $n$.

In both cases $f$ restricts to an automorphism of $H$. In particular, if in either of these cases $M$ is indecomposable, then $\text{End}(M)$ is a local ring.

Proposition 5. Let $R$ be a ring which is either left Noetherian or commutative and $M$ a left $R$-module which is a direct sum of submodules satisfying the d.c.c. By a further decomposition, these summands may be assumed indecomposable. Then any summand of $M$ is also a direct sum of submodules satisfying the d.c.c. and any direct sum decomposition of $M$ refines into a decomposition isomorphic to the original one.

Proof. It suffices to show that over such a ring an indecomposable module $N$ with d.c.c. satisfies condition (a) of the lemma. Let $N_i$ be the socle of $N$ and $N_{n+1}$ the inverse image in $N$ of the socle of $N/N_n$. We claim that $N$ is the union of the submodules $N_n$, and to show this it suffices to remark that any cyclic $R$-module satisfying the d.c.c. also satisfies the a.c.c., so that for $x \in N$, $Rx \subseteq N_n$ for some $n$. 


Proposition 6. Let $R$ be a left-Noetherian ring and $I$ a two-sided ideal generated by a finite number of elements in the center of $R$, such that $\bigcap_{n>0} I^n = 0$, $R$ is complete in the $I$-adic topology, and $R/I$ satisfies the d.c.c. Then if $M$ is a left $R$-module which is a direct sum of finitely generated modules, any summand of $M$ is again a direct sum of finitely generated modules. Further, any direct sum decomposition of $M$ refines into a direct sum of indecomposable, finitely generated modules, and this decomposition is unique up to isomorphism.

Proof. Clearly if $R/I$ satisfies the d.c.c., so does $R/I^n$ since $R$ is left-Noetherian. Hence if $N$ is finitely generated, $N/I^nN$ satisfies both chain conditions. Therefore it suffices to show that if $N$ is a finitely generated $R$-module, then $\bigcap_{n>0} I^n N = 0$, and $N$ is complete in the $I$-adic topology. This actually holds without the hypothesis that $R/I$ satisfies the d.c.c. Clearly a finitely generated module $L$ with $\bigcap_{n>0} I^n L = 0$ is complete in the $I$-adic topology. We represent $N$ as a quotient, $N = F/K$, where $F$ is free and finitely generated. It is enough to show that $\bigcap_{n>0} I^n N = 0$, and since $\bigcap_{n>0} I^n N$ is just the closure of 0 in the $I$-adic topology, this is equivalent to showing that $K$ is a closed submodule in the topology of $F$. Clearly $\bigcap_{n>0} I^n K = 0$, so $K$ is complete, and we need only show that the $I$-adic topology on $K$ agrees with the topology induced from the $I$-adic topology on $F$. This is a standard argument in commutative algebra (Bourbaki, Algèbre commutative, Chapter III, Corollary 1, p. 61) where $R$ is assumed to be commutative, and the fact that $I$ is generated by elements of the center of $R$ is enough to assure that the same arguments are valid in this case.

Corollary. If $P$ is a projective module over a ring satisfying the hypotheses of Proposition 6, then $P$ is a direct sum of indecomposable left ideals generated by idempotents and any two such decompositions of $P$ are isomorphic.

For examples, we remark that an Artinian ring satisfies these hypotheses trivially, and so does the group ring of a finite group over the $p$-adic integers.

Bibliography


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