SMOOTH BANACH ALGEBRAS[1]

I. N. SPATZ[2]

Introduction. The purpose of this note is to prove the theorem that follows. This theorem generalizes results of L. Ingelstam [2] and M. F. Smiley [4] for Hilbert algebras. It also provides a simpler proof of Corollary which was obtained by F. F. Bonsall and J. Duncan [1]. (This result was obtained before I was aware of the work of Bonsall and Duncan.)

Let $G(x, y)$ $(G_+(x, y), G_-(x, y))$ denote the limit as t approaches zero (from the right, from the left) of $(\|x+ty\|-\|x\|)/t$. For any of the properties of G used in the proof see Koethe [3].

THEOREM. *Let X be a real Banach algebra with identity e. $\|e\|=1$. If $G(e, s)$ exists for each singular element s of X, then X is a division algebra.*

PROOF. If $s \in X$ is singular, then $G(e, s)=0$ since $\|e+ts\| \geqq 1$ implies $G_+ \geqq 0$ and $G_- \leqq 0$.

It follows immediately that $e+s$ is invertible for $G(e, e+s)=G(e, e)+G(e, s)=1$.

Suppose $u \in X$ is invertible and $\|u\|=1$. Then $\|u+ts\| \leqq \|e+tu^{-1}s\|$. It follows immediately from the above inequality (and the fact that $u^{-1}s$ is singular if s is) that $G_+(u, s) \leqq G_+(e, u^{-1}s)=0$ and $G_-(u, s) \geqq G_-(e, u^{-1}s)=0$. But, in general, $G_+ \geqq G_-$. Thus $G(u, s)=0$.

If s is singular, then $u_\alpha=(e+\alpha s)/\|e+\alpha s\|$ (α a real number) is an invertible element of norm one. Hence

$$1 = G(u_\alpha, u_\alpha) = G(u_\alpha, e)/\|e + \alpha s\|.$$

Since $G(u_\alpha, e) \leqq \|u_\alpha\|\|e\|=1$,

$$1 \geqq \|e + \alpha s\| \geqq |\alpha| \|s\| - 1.$$

Evidently, this implies $s=0$.

COROLLARY. *If X is a smooth Banach algebra over the reals with identity e of norm one, then X is the reals, complexes or the real quaternions.*

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[2] The author held a National Science Foundation Traineeship during the time this research was done.

References

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3. G. Koethe, *Topologische Lineare Räume*, Springer-Verlag, Berlin, 1960, pp. 351–352.

4. M. F. Smiley, *Real Hilbert algebras with identity*, Proc. Amer. Math. Soc. **16** (1965), 440–441.

Brooklyn College