SMOOTH BANACH ALGEBRAS

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Introduction. The purpose of this note is to prove the theorem that follows. This theorem generalizes results of L. Ingelstam [2] and M. F. Smiley [4] for Hilbert algebras. It also provides a simpler proof of Corollary which was obtained by F. F. Bonsall and J. Duncan [1]. (This result was obtained before I was aware of the work of Bonsall and Duncan.)

Let $G(x, y)$ ($G_+(x, y), G_-(x, y)$) denote the limit as $t$ approaches zero (from the right, from the left) of $(||x+ty|| - ||x||)/t$. For any of the properties of $G$ used in the proof see Koethe [3].

Theorem. Let $X$ be a real Banach algebra with identity $e$. $||e|| = 1$. If $G(e, s)$ exists for each singular element $s$ of $X$, then $X$ is a division algebra.

Proof. If $s \in X$ is singular, then $G(e, s) = 0$ since $||e+ts|| \geq 1$ implies $G_+ \geq 0$ and $G_- \leq 0$.

It follows immediately that $e+s$ is invertible for $G(e, e+s) = G(e, e) + G(e, s) = 1$.

Suppose $u \in X$ is invertible and $||u|| = 1$. Then $||u+ts|| \leq ||e+tu^{-1}s||$. It follows immediately from the above inequality (and the fact that $u^{-1}s$ is singular if $s$ is) that $G_+(u, s) \leq G_+(e, u^{-1}s) = 0$ and $G_-(u, s) \geq G_-(e, u^{-1}s) = 0$. But, in general, $G_+ \geq G_-$. Thus $G(u, s) = 0$.

If $s$ is singular, then $u_s = (e+\alpha s)/||e+\alpha s||$ (a real number) is an invertible element of norm one. Hence

$$1 = G(u_s, u_s) = G(u_s, e)/||e+\alpha s||.$$ 

Since $G(u_s, e) \leq ||u_s|| ||e|| = 1,$

$$1 \geq ||e+\alpha s|| \geq \alpha ||s|| - 1.$$ 

Evidently, this implies $s = 0$.

Corollary. If $X$ is a smooth Banach algebra over the reals with identity $e$ of norm one, then $X$ is the reals, complexes or the real quaternions.

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1 Taken from a dissertation submitted to the Faculty of the Polytechnic Institute of Brooklyn in partial fulfillment of the requirements for the degree of Doctor of Philosophy (Mathematics), 1967.

2 The author held a National Science Foundation Traineeship during the time this research was done.
REFERENCES


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