CHARACTERIZATION OF CONNECTED 2-MANIFOLDS WITHOUT BOUNDARY WHICH HAVE FINITE DOMAIN RANK

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1. **Introduction.** Let M be a connected n-manifold, O an open subset. An n-manifold $m(O)$ is called a *generator* of O if O is an open monotone union of $m(O)$; that is, $O=\bigcup_{k=1}^{\infty}O_k$ where for all k, O_k is open in M, $O_k\subset O_{k+1}$, and O_k is homeomorphic to $m(O)$. As in the announcement of this result [3], a set G^* of connected n-manifolds is called a *set of generating domains* for M if for any proper domain (open, connected subset) D of M there is an element G of G^* which generates D. The *domain rank* of M, denoted by $DR(M)$ is defined by

$DR(M)=\text{g.l.b. }\{|G^*| : G^* \text{ is a set of generating domains for } M\}$, where $|G^*|$ denotes the cardinality of G^*. Note that if D is a domain of M, then $DR(D)\leqq DR(M)$. The main result established in this paper is that a connected 2-manifold without boundary has finite domain rank if and only if it is homeomorphic to some domain of a compact connected 2-manifold without boundary.

2. **Proof of the theorem.** Henceforth the term surface will mean a connected 2-manifold. If M is a surface without boundary, then M is called closed if M is compact; otherwise, M is called open. A compact surface will always mean a compact connected 2-manifold with boundary. If M is a surface, then bd M and int M will denote the boundary and interior of M respectively.

If M is a closed surface, then it is well known (see [2]) that M is either a 2-sphere or M has a topologically unique representation as the connected sum of n tori or n projective planes, $n\geqq 1$. As usual, the *genus* of M, denoted by $g(M)$, is defined to be 0 if M is a 2-sphere and is defined to be n if M is topologically the connected sum of n tori or n projective planes.

Let $\equiv$ denote topological equivalence. If M is a compact surface, then $M\equiv \text{Cl}(L-\bigcup_{k=1}^{p} C_k)$ where L is a closed surface and $\{C_k\}_{k=1}^{p}$ is a collection of pairwise disjoint 2-cells contained in L. As usual the *genus* of M, $g(M)$, is defined by $g(M)=g(L)$.

Henceforth let S^2 denote a fixed closed surface of genus 0, nT a fixed closed orientable surface of genus n, and nP a fixed closed nonorientable surface of genus n. For any surface M, let $\rho(M)=1$ if M is orientable; otherwise, let $\rho(M)=0$.

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DEFINITION 2.1. Let M be an open surface. The genus of M, denoted by $g(M)$, is defined by $g(M)=\sup\{g(K)\colon K$ is a compact surface, $\rho(K)=\rho(M)$, and $K\subset M\}$.

DEFINITION 2.2. Let M be an open surface, $\rho(M)=0$. The orientable genus of M, denoted by $\mathrm{og}(M)$, is defined by $\mathrm{og}(M)=\sup\{g(K)\colon K$ is a compact surface, $\rho(K)=1$, and $K\subset M\}$.

It is well known that the connected sum $T\#P\equiv P\#P\#P\equiv 3P$. Using this fact one can easily establish the following lemma.

LEMMA 2.3. *Let M be an open surface, $\rho(M)=0$. If $g(M)=n$, $1\leqq n<\infty$, then* $\mathrm{og}(M)=[(n-1)/2]$ ($[x]$ *denotes the largest integer k such that $k\leqq x$*).

LEMMA 2.4. *Let M and M' be open surfaces such that $M'\subset M$ and $\rho(M)=\rho(M')$. Then $g(M')\leqq g(M)$.*

PROOF. Follows immediately from Definition 2.1.

LEMMA 2.5. *Let M be an open surface, $g(M)<\infty$. If an open surface N generates M, then $g(M)=g(N)$ and $\rho(M)=\rho(N)$.*

PROOF. Since N generates M, $M=\bigcup_{j=1}^{\infty}N_j$, where for all $j\geqq 1$, $N_j\subset N_{j+1}$ and $N_j\equiv N$. Since $g(M)<\infty$, there exists a compact surface K such that $K\subset M$, $g(K)=g(M)$, and $\rho(K)=\rho(M)$. Therefore there exists an i such that $K\subset N_i\subset M$. It follows that $\rho(K)=\rho(N_i)=\rho(M)$ and Lemma 2.4 implies that $g(M)=g(K)\leqq g(N_i)\leqq g(M)$. Therefore $g(N)=g(N_i)=g(M)$ and $\rho(M)=\rho(N)$.

DEFINITION 2.6. If Q is a closed surface, let $Q(\infty)=Q-A$ where A is a subset of Q which is homeomorphic to $\{0\}\cup\{1/n\colon n\geqq 1\}$ of the real line.

LEMMA 2.7. *Let M be an open surface, $g(M)=n<\infty$, and let Q be a closed surface such that $g(M)=g(Q)$ and $\rho(M)=\rho(Q)$. Then $Q(\infty)$ is a generator of M.*

PROOF. Since $g(M)=g(Q)$ and $\rho(M)=\rho(Q)$, there is a sequence $\{K_i\}_{i=1}^{\infty}$ of compact surfaces such that $M=\bigcup_{i=1}^{\infty}K_i$ where for all $i\geqq 1$ $g(K_i)=g(Q)$, $\rho(K_i)=\rho(Q)$, and $K_i\subset \mathrm{int}\,K_{i+1}$. If N is a compact surface, let N^c denote the space obtained by attaching $\mathrm{bd}\,N\times[0,1]$ to N by the map $f\colon \mathrm{bd}\,N\times\{0\}\to N$ defined by $f(x,0)=x$, $x\in\mathrm{bd}\,N$, and consider N as embedded in the usual way in N^c. For any $i\geqq 1$, there is an embedding e_i of K_i^c into M such that $K_i\subset\mathrm{int}\,K_i'\subset K_i'\subset\mathrm{int}\,K_{i+1}$ where $e_i(K_i^c)=K_i'$ and $e_i(x)=x$ for all $x\in K_i$. Since $K_i'\equiv K_i'$, it follows that $K_i'\equiv\mathrm{Cl}(Q-\bigcup_{j=1}^{n(i)}C_j)$ where $\{C_j\}_{j=1}^{n(i)}$ is a disjoint collection of 2-cells contained in Q. Let A_i be a closed set in $\mathrm{int}\,K_i'-K_i$

which is homeomorphic to $A=\{0\}\cup\{1/n: n\geqq 1\}$. It is easily seen that $Q_i=(K_i'-(A_i\cup \text{bd } K_i'))\equiv Q(\infty)$. Since $K_i\subset Q_i\subset K_{i+1}$, it follows that $Q(\infty)$ generates M.

THEOREM 2.8. *If M is an open surface, $g(M)=n<\infty$, then $DR(M)$ $=n+[(n-1)/2](1-\rho(M))+1$.*

PROOF. (i) Suppose first that $\rho(M)=1$. If D is a domain of M, then $\rho(D)=1$ and it follows from Lemma 2.4 that $0\leqq g(D)=k\leqq n$. Therefore it follows from Lemma 2.7 that D is generated by $S^2(\infty)$ if $k=0$ and by $kT(\infty)$ if $1\leqq k\leqq n$. Therefore $\{S^2(\infty), T(\infty), \cdots, nT(\infty)\}$ is a set of generating domains for M and so $DR(M)\leqq n+1$.

If $0\leqq k\leqq n$, then there is a domain D_k of M such that $g(D_k)=k$, and $\rho(D_k)=1$. If Q_k generates D_k, then it follows from Lemma 2.5 that $g(D_k)=g(Q_k)$ and $\rho(Q_k)=\rho(D_k)$. This implies that $DR(M)$ $\geqq n+1$. Therefore $DR(M)=n+[(n-1)/2](1-\rho(M))+1$.

(ii) Now suppose that $\rho(M)=0$ and that $g(M)=n$. If D is any domain of M, then it follows from Lemma 2.3 and Lemma 2.4 that either $\rho(D)=0$ and $1\leqq g(D)\leqq g(M)$ or $\rho(D)=1$ and $0\leqq g(D)$ $\leqq [(n-1)/2]$. As above it follows from Lemma 2.7 that $\{P(\infty), \cdots, nP(\infty), S^2(\infty), \cdots, [(n-1)/2]T(\infty)\}$ is a set of generating domains for M and so $DR(M)\leqq n+[(n-1)/2](1-\rho(M))+1$.

If $1\leqq k\leqq n$, then there is a domain D_k of M such that $\rho(D_k)=0$ and $g(D_k)=k$. If Q_k generates D_k, then it follows from Lemma 2.5 that $\rho(Q_k)=\rho(D_k)$ and $g(Q_k)=g(D_k)$. If $0\leqq j\leqq [(n-1)/2]$, then there is a domain D_j of M such that $\rho(D_j)=1$ and $g(D_j)=j$. If Q_j generates D_j, then again it follows from Lemma 2.5 that $\rho(Q_j)=\rho(D_j)$ and $g(Q_j)=g(D_j)$. Hence $DR(M)\geqq n+[(n-1)/2](1-\rho(M))+1$ and therefore equality holds.

COROLLARY 2.9. *An open surface M has finite domain rank if and only if M is homeomorphic to a domain of a closed surface N.*

PROOF. Suppose that N is a closed surface of genus n and that $M\equiv D$, D a domain of N. If $\rho(N)=1$, then $\rho(D)=1$ and $g(D)=k$, $0\leqq k\leqq n$. Therefore it follows from Theorem 2.8 that $DR(D)=DR(M)$ $=k+1$. If $\rho(N)=0$, then either $\rho(D)=1$ and $g(D)=k$, $0\leqq k$ $\leqq [(n-1)/2]$ or $\rho(D)=0$ and $g(D)=k$, $1\leqq k\leqq n$. In either case it follows from Theorem 2.8 that $DR(D)=k+[(k-1)/2](1-\rho(D))+1$.

Now suppose that $DR(M)<\infty$. Then it follows from Lemma 2.5 that $g(M)<\infty$. Let $g(M)=n$ and let N be a closed surface such that $\rho(N)=\rho(M)$ and $g(N)=g(M)=n$. Since $g(M)=n$, there is a sequence $\{K_i\}_{i=1}^{\infty}$ of compact surfaces such that $M=\bigcup_{k=1}^{\infty} K_i$ where $\rho(K_i)$

$=\rho(M)$, $g(K_i)=g(N)$, and $K_i \subset \text{int } K_{i+1}$ for all $i \geqq 1$. Since for each $i \geqq 1$, $K_i \equiv \text{Cl}(N - \bigcup_{j=1}^{q(i)} C_j^i)$ where $\{C_j^i\}_{j=1}^{q(i)}$ is a disjoint collection of 2-cells in N, it follows that for all $i \geqq 1$, $K_{i+1} - \text{int } K_i$ is a finite disjoint collection of compact surfaces of genus zero. Therefore any embedding f_i of K_i into N can be extended to an embedding f_{i+1} of K_{i+1} into N. Therefore there exists a sequence $\{f_i\}_{i=1}^{\infty}$ of embeddings into N such that for $i \geqq 1$, f_i is an embedding of K_i into N and $f_{i+1} | K_i = f_i$. Define $f\colon M \rightarrow N$ by $f_i(x)$ if $x \in K_i$. Then f is an embedding of M into N.

It is possible to use the results of [2] to prove Lemma 2.7 and Corollary 2.9. However the author has chosen to establish them by more elementary methods.

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References

1. W. Massey, *Algebraic topology, An Introduction*, Harcourt, Brace and World, New York, 1967.

2. I. Richards, *On the classification of noncompact surfaces*, Trans. Amer. Math. Soc. **106** (1963), 259–269.

3. R. J. Tondra, *Characterization of connected 2-manifolds without boundary which have finite domain rank*, Abstract 68T-G5, Notices Amer. Math. Soc. **15** (1968), 810.

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