ON A THEOREM OF OREY

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The starting point of this paper is the following theorem of Orey [1]: Let $T$ be a linear operator on $L_1$ of the integers (i.e., $T$ acts on absolutely convergent series). Assume

(a) $T$ is positive (i.e., if $f$ is a nonnegative function on the integers then so is $Tf$),
(b) $\|T\|_1 \leq 1$ (i.e., $\int Tfdu \leq \int fdu$ where $u$ is the counting measure on the integers),
(c) $f \geq 0, f \neq 0$ implies $\sum_{i=1}^\infty Tf = \infty$ everywhere,
(d) for each pair of integers $i, j$ there is an $n_0(i, j)$ such that if $\delta_i$ is the function that is 1 on $i$ and 0 elsewhere then $T^n \delta_i(j) > 0$ for all $n > n_0$.

Under these conditions if $f$ is a function such that $\int fdu = 0$ (and $\int f|f|du < \infty$) then $\lim_{n \to \infty} \int T^n f du = 0$.

Sucheston began the study of operators where condition (b) is replaced by (b') $\sup_n \|T\|_1 < \infty$. He shows, using Banach limits, that either $\lim_{n \to \infty} \int T^n f du = 0$ for all $f$ in $L_1$ or we can find a bounded function $\varepsilon$, strictly greater than 0, such that $T^* \varepsilon = \varepsilon$. ($T^*$ is the adjoint of $T$.) (See Theorem 1.2 in part 1 of [2]). He conjectured in a letter to the author that if $\int |f|du < \infty$ and $\int f \cdot edu = 0$ then $\lim_{n \to \infty} \int T^n f du = 0$. The purpose of this note is to prove that conjecture.

Let $v$ be the measure induced on the integers by $\varepsilon$. $(\int fdu = \int f \cdot edu.)$ Then $\int f dv = \int Tf du$ and hence Orey's theorem shows that if $\int f dv = 0$ then $\lim_{n \to \infty} \int T^n f dv = 0$. What needs to be shown is that $\lim_{n \to \infty} \int T^n f du = 0$. ($\varepsilon$ may not be bounded away from 0.)

**LEMMA.** Let $u$ and $v$ be measures on the integers. Let $T$ be a linear operator satisfying (a), (c), (b') where $\|T\|_1$ is computed with respect to $u$, and (b) where $\|T\|_1$ is computed with respect to $v$. If $\lim_{n \to \infty} \int T^n f dv = 0$ then $\lim_{n \to \infty} \int T^n f du = 0$.

We will now give a proof of the above lemma. We will start with the following:

**DEFINITION.** We will call a function $\tilde{g}$ a version of $g$ ($g \geq 0$) if for some integer $n$, $\tilde{g}$ is obtained from $g$ by the following process: Let $g = g_1 + g_1$ where $g_1 \geq 0$ and $\tilde{g}_1 \geq 0$ (for the rest of this definition all functions will be nonnegative). Let $T\tilde{g}_1 = \tilde{g}_2 + \tilde{g}_2$ and let $T\tilde{g}_i = g_{i+1} + g_{i+1}$. Let

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\[ g = \left( \sum_{i=1}^{n} g_i \right) + \tilde{g}_n. \]

**Remarks.** (A) \( T^{K-1}g = \sum_{i=1}^{n} T^{K-1}g_i + T^{K-n}\tilde{g}_n \) for \( K > n \).

(B) If we replace any \( g_i \) (or \( \tilde{g}_n \)) by a version of \( g_i \) (or \( \tilde{g}_n \)) we still have a version of \( g \).

(C) \( \int gdv = \int \tilde{g}dv. \)

Let \( f = g - h \), \( g \geq 0 \), \( h \geq 0 \). It is easy to see that without loss of generality we could assume that both \( g \) and \( h \) are 0 for all but one integer.

1. If \( \lim_{t \to \infty} \left( \lim_{t \to \infty} \int |T^t(g-h)|dv \right) = 0 \) then we are finished by (b'). We can therefore assume that (1) is false, i.e., there is an \( \alpha > 0 \) such that the left side of (1) is \( > \alpha \). This implies

2. given \( \epsilon \) we can find an \( n \) such that \( T^ng = r + s \), \( r \geq 0 \), \( s \geq 0 \), \( \int rdv \geq (1-\epsilon) \int \tilde{g}dv \), \( \lim_{t \to \infty} \int T^t|s|dv > \alpha/2 \) (\( r = \inf(T^ng, T^nh) \)). (2) is true for \( g \) or \( h \) so we assume it true for \( g \).

3. We can find a version \( r \) of \( r \), \( r = r_1 + r_2 + \cdots + r_m + r_m \), such that \( r_i \), \( i = 1, \ldots, m \), is 0 or a multiple of \( g \), and \( \int r_i dv < \epsilon \int rdv \). ([a], [b] with \( u \) replaced by \( v \), and [c] imply [3] by a standard argument which goes as follows: Let \( j \) be the integer at which \( g \) is non 0. Let \( r_{i+1} \) be the part of \( T^r_i \) that has support at \( j \). It is enough to show that \( \lim_{t \to \infty} \int r_i = 0 \) and we may assume that the support of \( r \) is one point, \( l \). Let \( \beta(l) = \lim_{t \to \infty} \int r_i \) when the support of \( r = l \). \( \int \sum_{i=1}^{n} r_i = 1 - \beta(l) \). Since \( (\sum_{i=1}^{n} T\delta_i) \) \( (j) = \infty \), it is easily seen that \( \beta(j) = 0 \). Since \( (\sum_{i=1}^{n} T\delta_j) \) \( (l) \neq 0 \), \( \beta(j) = 0 \) implies that \( \beta(l) = 0 \).]

4. (2) and (3) imply that we can get a version \( \tilde{g} \) of \( g \) such that \( \tilde{g} \geq \gamma g + s \) where \( \gamma > (1-\epsilon)2. \) (\( s \) is defined in (2).)

5. It follows from (4), since \( \epsilon \) could be made arbitrarily small, that for any large number \( M \) we could find a version \( \tilde{g} \) of \( g \) such that \( \tilde{g} > Ms' \) where \( s' \) has the property that \( \lim_{t \to \infty} \int T^t|s'|dv > \alpha \) large enough.

6. It follows from (5) that \( \int T^k \tilde{g}dv > \frac{1}{2} M \alpha \) for all \( K \). Using remark (A), if \( K \) is large enough we could write \( T^k \tilde{g} = \sum_{i=1}^{n} T^i(T^k-\delta_i) + T^n(T^k-\delta_n) \) where \( \tilde{g} = \sum_{i=1}^{n} \delta_i + \tilde{g}_n \) and \( T^k g = \sum_{i=1}^{n} T^k-\delta_i + T^k-\delta_n \).

If \( \sup T^t = B \) this shows that \( \int T^k \tilde{g}dv > \frac{1}{2} M \alpha (B)^{-1} \), but since \( M \) could be arbitrarily large this contradicts (b').

**Another Proof of Orey's Theorem.** The same kind of argument as the one above will give another proof of Orey's theorem, and we will now indicate how to do this. (We will assume that (a)--(d) are true for the measure \( u \), and we will forget about \( v \).)

1. We could, as before, assume that \( f = g - h \) where \( g \geq 0 \), \( h \geq 0 \) and \( g \) and \( h \) have only 1 point in their support (i.e., \( g = \delta_x \), \( h = \delta_y \)).
(2') It is enough to show that \( \lim_{n \to \infty} \int |T^n g - T^{n+1} g| \, du = 0 \). To see this note that (3) tells us we can get a version \( h \) of \( h \) such that 
\[
h = n_1 + \cdots + n_n.
\]
where each \( n_i \) is 0 or a multiple of \( g \) and \( \int h \, du < \epsilon \). (2') now follows from Remark A.

(3') Given \( M \) we can find \( \alpha > 0 \), \( n > N \), and a version \( \bar{g} \) of \( g \) such that 
\[
\bar{g} = \sum_{i=1}^n g_i + g_n,
\]
where \( g_i = 0 \) for \( i < n - M \) and \( g_i = \alpha g \) for \( n - M \leq i \leq n \). This follows easily from (d).

(4') Given \( \epsilon > 0 \) we can find a version \( \bar{g} \) of \( g \) such that 
\[
\bar{g} = \sum_{i=1}^n g_i + g_n,
\]
where each \( g_i = \sum_{j=1}^m g_{i,j} \) and for fixed \( j \) all the \( g_{i,j} \) are 0 except for a block of \( M \) consecutive \( i \) and for these \( i \) all the \( g_{i,j} \) are equal to the same multiple of \( g \). To get (4') we use (3'). Then apply (3) to get a version \( \bar{g} \) of \( g_n \) where \( \bar{g} = \sum_{i=1}^n g_i + g_n, \int \bar{g} \, du \) is very small and each \( g_i \) is a multiple of \( g \). We next apply this whole process to each \( g_i \), etc.

(5') Let \( \bar{g} \) be the version of \( g \) constructed in (4'). By Remark A, 
\[
T^{K-1} \bar{g} = \sum_{i=1}^m T^{K-1} g_i + T^{K-m} \bar{g}_m
\]
and hence 
\[
T^{K-1} \bar{g} = \sum_{j=1}^l \left( \sum_{i=1}^m T^{K-i} g_{i,j} \right) + T^{K-m} \bar{g}_m,
\]
\[
T^K \bar{g} = \sum_{j=1}^l \left( \sum_{i=1}^m T^{K-i+1} g_{i,j} \right) + T^{K-m+1} \bar{g}_m
\]
\[
= \sum_{j=1}^l \left( \sum_{i=1}^m T^{K-i} g_{i-1,j} \right) + T^{K-m+1} \bar{g}_m.
\]

We now use the fact that for fixed \( j \), \( g_{i-1,j} = g_{i,j} \), except for 2 values of \( i \). This shows that 
\[
\int \left| T^K \bar{g} - T^{K-1} \bar{g} \right| \, du \leq \frac{2}{M} \int \bar{g} \, du + 2\epsilon.
\]
This gives (2') and therefore Orey's theorem.

Bibliography

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