THE TOWER THEOREM FOR FINITE GROUPS

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This note aims to give a simple proof of Wielandt’s theorem that the tower of automorphisms of a finite group with center 1 ends after a finite number of steps (cf. [2] and [4, p. 245]).

Our notation is as follows: $B < G$, $A < G$, denote respectively that $B$ is invariant and $A$ subvariant in $G$; $C_B(A)$ and $\mathcal{N}_B(A)$ denote respectively the centralizer and normalizer of $A$ in $B$; $\mathfrak{A}(A)$ is the automorphism group of $A$, $|A|$ the cardinality of $A$, and $A^\omega$ is the smallest normal subgroup of $A$ with $A/A^\omega$ nilpotent. All groups are finite.

Our proof of the tower theorem is based on the following facts.

(1) If $C_B(A) = 1$ then $C_{\mathcal{N}(A)}(A) = 1$.

(2) If $A = A_0 < A_1 < \cdots < A_r = G$ and if for $i = 0, 1, \cdots, r - 1$, $C_{A_{i+1}}(A_i) = 1$, then $C_{\mathcal{N}(A)}(A) = 1$ (cf [2, p. 244]).

(3) If $A < G$ and if $C_{\mathcal{N}(A)}(A) = 1$, then $C_{\mathcal{N}(A^\omega)}(A) \leq A^\omega$ (cf. [1]).

(4) If $A < G$ then $A^\omega B = B A^\omega$ for any $B < G$ (cf [3]).

As a corollary of (4) we have the following

(4*) If $H < G$ and $A < H$, then $H A^\omega B = B H A^\omega$.

We also will want the following readily proven fact.

(5) If $A < B$ and $A < B$ is simple then $A < B$.

With these results we can now prove the following result and then the tower theorem will be a direct consequence.

Theorem. Let $A < G$ and suppose that $C_{\mathcal{N}(A)}(A) = 1$, then $|G|$ is bounded in terms of $|A^\omega|$.

Proof. Let $s_i$ denote the set of simple subvariant subgroups $B_i$ of $G$ and for $i = 1, 2, \cdots$, let $s_{i+1}$ denote the set of subinvariant subgroups $B_{i+1}$ of $G$ such that $B_{i+1}$ contains as a normal subgroup a $B_i$ of $s_i$ with $B_{i+1}/B_i$ simple. For each $i$, all the subgroups $B_i$ of $s_i$ generate a normal subgroup $H_i$ of $G$ and we let $K_i$ denote $H_i A^\omega$ (with $K_0$ denoting $A^\omega$ and $K_0$ denoting 1). Since $C_{\mathcal{N}(A)}(A) = 1$, $A^\omega \neq 1$ and hence $H_0 \neq K_0$. Let $n$ be minimal so that $H_n = K_n$ ($n$ is at most the length of a composition series of $A^\omega$). Then for $i = 0, \cdots, n - 1$, $K_i < K_i B_{i+1}$ for any $B_{i+1} \in s_{i+1}$ by (5). Hence $B_{i+1}$ and consequently $H_{i+1} \leq \mathcal{N}(K_i)$. Then in view of (3) and the fact that $K_i = H_i A^\omega$, $|K_i| \leq |A^\omega| \cdot |\mathfrak{A}(K_i)|$ for $i = 1, 2, \cdots$ and hence in particular $|K_n|$ is bounded in terms of $|A^\omega|$. But $K_n = H_n$ is normal in $G$ and since

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it follows that \(|G| \leq |A^a| \cdot |\mathcal{A}(K_n)|\). This proves the theorem.

It follows from the theorem with (1) and (2) that the tower of automorphisms of a finite group with center 1 is finite.

References


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