A NONOSCILLATION THEOREM FOR A NONLINEAR SECOND ORDER DIFFERENTIAL EQUATION

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In this paper we consider the real-valued solutions of the equation

\[(1) \quad y'' + q(t)y = 0\]

where \(q(t) \geq 0\) and continuous on some half line \([a, \infty)\) and \(\gamma\) satisfies \(0 < \gamma = p/q < 1\) where \(p, q\) are odd integers. Our purpose is to give conditions under which all solutions of \(1\) are nonoscillatory. The result we give is similar to that given by Atkinson \([1]\) for the case \(\gamma > 1\) but the proof is different.

The restriction to \(\gamma = p/q\) where \(p\) and \(q\) are odd is significant. For example, if \(q\) is even and \(p\) odd, then oscillatory solutions are not real-valued. If \(p\) is even and \(q\) odd, then all nonzero solutions are trivially nonoscillatory. Similar problems arise if \(\gamma\) is irrational.

\[\text{We begin with some definitions and basic facts. A solution of } (1) \text{ is said to be extendable (continuable) if it exists on some half line } [b, \infty). \text{ Since } 0 < \gamma < 1, \text{ all solutions of } (1) \text{ are extendable. This follows from a theorem of Wintner } [3, \text{ p. 29}]. \text{ A nontrivial solution of } (1) \text{ is called oscillatory if it has arbitrarily large zeros. Otherwise, a solution is called nonoscillatory, i.e., if it is of one sign for large } t. \text{ Since } \gamma \text{ is restricted to be odd, solutions with real initial conditions are real-valued and the negative of a solution is again a solution.}\]

\[\text{For the sake of completeness we state some related results. Lichko and Švec } [5] \text{ have shown that all solutions of } (1) \text{ are oscillatory if and only if } \int_{a}^{\infty} s^q(s)ds = \infty. \text{ Belohorec } [2] \text{ has shown the following. If there exists a number } \beta, 0 < \beta < (1 - \gamma)/2, \text{ such that } f(t)\beta 1/(2+\beta) \uparrow K_1 < \infty, \text{ then all nontrivial solutions of } (1) \text{ are nonoscillatory. If } f(t)\beta 1/(2+\beta) \downarrow K_2 > 0, \text{ then } (1) \text{ has both (nontrivial) oscillatory and nonoscillatory solutions. For similar results pertaining to the case } 1 < \gamma, \text{ see } [1] \text{ and } [4].\]

\[\text{We can now state our major result. Its proof will be preceded by three lemmas.}\]

**Theorem.** If \(q(t) \in C'[a, \infty), q(t) > 0 \text{ and } q'(t) \leq 0 \text{ for } t \geq a \text{ and if } \int_{a}^{\infty} s^q(s)ds < \infty, \text{ then } (1) \text{ has no oscillatory solutions.}\]

**Lemma 1.** Suppose that \(\int_{a}^{\infty} s^q(s)ds < \infty \) and let \(K > 0\) be given. Then

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there is a $t_0 \geq a$ and a solution $y(t)$ of (1) defined on $[t_0, \infty)$ such that $K/2 \leq y(t) \leq K$ for $t \geq t_0$ and $\lim_{t \to \infty} y(t) = K$.

**Proof.** Our proof is a modification of a proof given in [1]. Consider the integral equation

\[
(2) \quad \psi(t) = K - \int_t^\infty (s - t)q(s)(\psi(s))^\gamma ds.
\]

Let $t_0$ be such that

\[
\int_{t_0}^\infty (s - t_0)q(s)ds < \min\{(K^{1-\gamma})/2, [\gamma(2/K)^{1-\gamma}]^{-1}\}.
\]

To prove the lemma it suffices to show that (2) has a solution $\psi(t)$ such that $K/2 \leq \psi(t) \leq K$.

Let $\psi_0(t) = K$, $t \geq t_0$, and

\[
\psi_{n+1}(t) = K - \int_t^\infty (s - t)q(s)(\psi_n(s))^\gamma ds, \quad t \geq t_0.
\]

Then $K/2 \leq \psi_n(t) \leq K$ for $t \geq t_0$. Note that $F(\psi) = \psi^\gamma$ satisfies a Lipschitz condition for $K/2 \leq \psi \leq K$ with Lipschitz constant $\gamma(2/K)^{1-\gamma}$. Therefore

\[
| (\psi_{n+1}(t))^\gamma - (\psi_n(t))^\gamma | \leq \gamma(2/K)^{1-\gamma} | \psi_{n+1}(t) - \psi_n(t) |
\]

for $t \geq t_0$ and

\[
| \psi_{n+1}(t) - \psi_n(t) | \leq \gamma(2/K)^{1-\gamma} \max_{t \geq t_0} | \psi_n(t) - \psi_{n-1}(t) | \int_t^\infty (s - t)q(s)ds
\]

also for $t \geq t_0$. This shows that $\psi_n(t) \to \psi(t)$ uniformly on $[t_0, \infty)$ and hence $\psi(t)$ is a solution of (2) satisfying $K/2 \leq \psi(t) \leq K$ for $t \geq t_0$.

**Lemma 2.** Suppose that $q(t) \in C([a, \infty)$, $q(t) > 0$ and $q'(t) \leq 0$ for $t \geq a$. Let $y(t)$ be a nontrivial oscillatory solution of (1). Let $\{t_n\}$ be a sequence of consecutive relative maxima of $|y(t)|$ such that $n > m \Rightarrow t_n > t_m$. Then $|y(t_n)|$ is nondecreasing as $n$ increases and $\lim_{n \to \infty} t_n = \infty$.

**Proof.** Multiply (1) by $y'(t)/q(t)$ and integrate from $t_n$ to $t_{n+1}$ to obtain

\[
\int_{t_n}^{t_{n+1}} \frac{(y'(s))^2}{2} \frac{q'(s)}{q(s)^2} ds + \frac{(y(t_{n+1}))^{\gamma+1}}{\gamma + 1} - \frac{(y(t_n))^{\gamma+1}}{\gamma + 1} = 0.
\]

Since $q'(t) \leq 0$, we get $|y(t_{n+1})| \geq |y(t_n)|$.

Note that $\lim_{n \to \infty} t_n = \infty$ is not immediate because global unique-
ness for initial value problems does not hold in the case \( \gamma < 1 \). Suppose that \( \lim_{n \to \infty} t_n = t^* < \infty \). Since \( |y(t)| \) is increasing at its relative maxima, we can apply the mean value theorem to get a sequence \( \{ s_n \} \to t^* \) such that \( \lim_{n \to \infty} |y'(s_n)| = \infty \). But this contradicts the fact that \( y(t) \) exists on \([a, \infty)\).

**Lemma 3.** Let \( u(t), v(t), w(t) \) be solutions of (1) satisfying \( 0 \leq u(t) \leq v(t) \leq w(t) \) for \( t' \leq t \leq t'' \). Define \( \phi(t) \) by

\[
\phi(t) = (w - v)(v' - u') - (v - u)(w' - v').
\]

Then \( \phi(t') \geq \phi(t'') \).

**Proof.** The statement and proof of this lemma are adapted from Lemma 1 of [6]. Note that in our case

\[
(v' - u')(v - u) \geq (w' - u')(v - u).
\]

**Proof of Theorem.** Suppose to the contrary that \( y_1(t) \) is an oscillatory solution of (1). Let \( \{ t_n \} \) be the sequence of consecutive relative maxima of \( |y_1(t)| \). Then \( \lim_{n \to \infty} t_n = \infty \) and \( 0 < \lim_{n \to \infty} |y_1(t_n)| = L \leq \infty \) by Lemma 2.

Let \( 0 < K < L \) and let \( y_2(t) \) be a solution of (1) such that \( y_2(t) \uparrow K \) as \( t \to \infty \) (by Lemma 1). Then we can find two points \( t', t'' \) such that the following situation occurs: \( 0 < y_1(t') = y_2(t') \), \( 0 < y_1(t'') = y_2(t'') \), and \( 0 < y_2(t) < y_1(t) \) for \( t' < t < t'' \). If we now set \( u(t) = 0, v(t) = y_2(t), \) and \( w(t) = y_1(t) \), we see that \( \phi(t') < \phi(t'') \) (\( \phi(t) \) is defined in Lemma 3). But this contradicts Lemma 3. This proves the theorem.

**Remark.** The question arises as to whether the conditions \( q(t) > 0, q'(t) \geq 0 \) are necessary in the theorem. We conjecture that the weaker condition \( q(t) \geq 0 \) is not sufficient. However, this weaker condition is sufficient for the following corollary.

**Corollary.** If \( \int_{a}^{\infty} q(s)ds < \infty \) and if \( y(t) \) is an oscillatory solution of (1), then \( \lim_{t \to \infty} y(t) = \lim_{t \to \infty} y'(t) = 0 \).

**Proof.** The fact that \( \lim_{t \to \infty} y(t) = 0 \) follows from the proof of the theorem. Given \( \epsilon > 0 \) pick \( t_0 \) so large that \( \int_{t_0}^{\infty} q(s)ds < 1 \) and \( |y(t)| \leq \epsilon \) for \( t \geq t_0 \). Since \( y(t) \) is oscillatory we may suppose that \( y'(t_0) = 0 \). Therefore, by integrating (1) from \( t_0 \) to \( t \) we get

\[
|y'(t)| \leq \epsilon \int_{t_0}^{t} q(s) < \epsilon, \quad t \geq t_0.
\]

Since \( \epsilon \) is arbitrary it follows that \( \lim_{t \to \infty} y'(t) = 0 \) (see also [2, Theorem 2]).
References

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