A METRIZATION THEOREM FOR LINEARLY ORDERABLE SPACES

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A topological space X is *linearly orderable* if there is a linear ordering of the set X whose open interval topology coincides with the topology of X. It is known that if a linearly orderable space is semimetrizable then it is, in fact, metrizable [1]. We will use this fact to give a particularly simple metrization theorem for linearly orderable spaces, namely that a linearly orderable space is metrizable if and only if it has a G_δ diagonal. This is an interesting analogue of the well-known metrization theorem which states that a compact Hausdorff space is metrizable if it has a G_δ diagonal.

THEOREM. *A linearly orderable space with a G_δ diagonal is metrizable.*

PROOF. Suppose that X is linearly orderable and that the set $\Delta = \{(x, x) \mid x \in X\}$ is a G_δ in the space $X \times X$, say $\Delta = \bigcap_{n=1}^{\infty} W(n)$. We may assume that $W(n+1) \subseteq W(n)$ for each $n \geqq 1$. For each $x \in X$ and each $n \geqq 1$, there is an open interval $g(n, x)$ in X such that $(x, x) \in g(n, x) \times g(n, x) \subseteq W(n)$ and such that $g(n+1, x) \subseteq g(n, x)$. It is easily verified that the collection $\{g(n, x) \mid n \geqq 1\}$ is a local base at x for each $x \in X$.

Suppose that $y \in X$ and that $<x(n)>$ is a sequence in X such that $y \in g(n, x(n))$ for each $n \geqq 1$. Clearly, if $z \in \bigcap_{n=1}^{\infty} g(n, x(n))$ then $(z, y) \in \Delta$. Therefore, if $r < y < s$, we can find an integer N such that $r, s \notin g(n, x(n))$ whenever $n \geqq N$. Therefore, $x(n) \in]r, s[$ for $n \geqq N$. Hence $<x(n)>$ converges to y. It now follows from [2, Theorem 3.2] that X is semimetrizable. Therefore, X is metrizable [1, Theorem 4.25].

REMARK 1. Note that our theorem cannot be proved by applying the metrization theorem for compact Hausdorff spaces mentioned above to the space X^+, the natural order compactification of X, since it may happen that X has a G_δ diagonal while X^+ does not. Consider, for example, the "interior points" $\{(x, y) \mid 0 \leqq x \leqq 1 \text{ and } 0 < y < 1\}$ of the unit square with the lexicographic order topology. This space has a G_δ diagonal and is metrizable, but since it is not separable, its order compactification cannot have a G_δ diagonal.

REMARK 2. It should be pointed out that a linearly orderable space X in which closed sets are G_δ's need not be metrizable, even

Received by the editors February 20, 1969.

if X is compact. The space $[0, 1]\times\{0, 1\}$ with the lexicographic order topology provides an example.

REMARK 3. As an application of our metrization theorem, we observe that the Sorgenfrey line (i.e., the real line with the left half open interval topology) is not linearly orderable since it is nonmetrizable and has a G_δ diagonal.

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