A NOTE ON PUNCTURED DISKS IN A 2-MANIFOLD

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Let $D_n$ be a punctured disk with $n - 1$ holes and let $M$ be a separable 2-manifold. Let $f: D_n \to M$ be a map which is an embedding of the boundary $\partial D_n$ of $D_n$.

**Theorem.** Some subfamily of the family

$$C_1 \cup \cdots \cup C_n = f(\partial D_n)$$

of simple loops contains $C_1$ and bounds an embedded punctured disk in $M$.

This was proved in [1] for $n = 1$, and in [2] for $n = 1, 2$ when $M$ is closed and orientable.

**Proof.** We assume $n > 1$. Observe that $C_1 \cup \cdots \cup C_n$ is a boundary in the mod 2 homology of $M$. Thus any $C_i$ is homologous to $\bigcup_{j \neq i} C_j$. If $d$ were a generating loop of a handle or a Mobius band in $M$ not meeting any other $C_j$, then it would generate a direct summand of the first homology group with $\bigcup_{j \neq i} C_j$ in the complementary summand. In particular each $d$ is two sided.

Form a manifold $N$ by cutting $M$ apart along the curves $C_i$, $i = 2, \ldots, n$ (but not $C_1$) and attaching disks $E_i, E'_i$ to the resulting boundary components. Form a space $\tilde{N}$ by identifying $E_i$ with $E'_i$ for $i = 2, \ldots, n$. Then $C_1$ is null homotopic in $\tilde{N}$ since $C_2, \ldots, C_n$ bound disks. We show that $C_1$ is null homotopic in $N$.

Suppose not and consider what happens with the fundamental groups as we paste together one pair of disks at a time. If we bring together two components of $N$ then we get the free product of their fundamental groups. If we paste together disks on the same component then we add a new generator and no relations. In neither case does $C_1$ become null homotopic.

By [1], $C_1$ bounds a disk $D$ in $N$. Remove from $D$ any of the $E_i$'s which it contains and take the closure of the resulting subset of $M$. If at most one of each pair $E_i, E'_i$ were contained in $D$, then the closure would be the desired punctured disk. If some pair $E_i, E'_i$ were contained in $D$, then the closure would include a handle of $M$ having $C_i$ as a generating loop and not meeting any other $C_j$. This contradiction finishes the proof.

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References