RIEMANNIAN MANIFOLDS OF CONSTANT $k$-NULLITY

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1. Introduction. The purpose of this note is to derive curvature conditions that will guarantee the existence of a product structure for a Riemannian manifold of constant $k$-nullity. The proof is modeled after similar theorems for Riemannian and Kähler manifolds of constant nullity [5], [6]. Nullity was defined by Chern and Kuiper [1]. Ōtsuki defined the concept of nullity relative to a constant $k$, so that nullity became the special case $k=0$ [4]. A definition in terms of vectors was given by Gray, who also shortened the name to $k$-nullity [2].

2. Definitions and the main theorem. Let $M_m$ denote the tangent space to the Riemannian manifold $M$ at the point $m$, and let $R_{xv}$ denote the curvature transformation associated with $x, y \in M_m$.

Definition. Let $B_{xyz} = R_{xyz} - k \langle (x, z)y - (y, z)x \rangle$, where $x, y, z \in M_m$ and $k$ is a constant.

Then $B$ is a tensor of the same type as $R$, and $B$ possesses the symmetries of $R$, [2].

Definition. Let $N_k(m) = \{z \in M_m : B_{xyz} = 0 \text{ for all } x, y \in M_m \}$. $N_k(m)$ is called the $k$-nullity space at $m$. The dimension $p(m)$ of $N_k(m)$ is the $k$-nullity at $m$. The conullity space $C_k(m)$ is the orthogonal complement to the nullity space at $m$. Elements of $C_k(m)$ are called conullity vectors. A conullity plane is a plane spanned by conullity vectors.

Theorem. Let $M^n$ be a complete, connected, and simply connected $C^\infty$ Riemannian manifold of constant $k$-nullity $\mu$, where $0 < \mu \leq n - 3$. If $n-\mu$ is odd and the sectional curvatures of all conullity planes are unequal to $k$, then $M^n$ is a direct metric product, $M^n = K^\mu \times C^{n-\mu}$, where $K^\mu$ and $C^{n-\mu}$ are complete, and $K^\mu$ has constant curvature $k$.

3. Proof of the theorem. If $\mu$ is constant and positive, the distribution of $k$-nullity spaces is integrable, and the integral manifolds are complete submanifolds of $M^n$ of constant curvature $k$, [2]. Any one of these integral manifolds provides one factor for a product structure of $M^n$.

Definition. For each $u \in N_k(m)$ and $x \in C_k(m)$, let $T_u(x) = P(\nabla_x U)$,
where \( P \) is the projection of \( M^n \) into \( C_k(m) \) and \( U \) is any nullity extension of \( u \).

\( T_u \) is a well-defined linear operator on \( C_k(m) \), called a conullity operator [6]. The nonvanishing of the conullity operators represents the obstruction to the existence of a product structure for \( M^n \), for if each conullity operator is zero, we can apply DeRham’s decomposition theorem to obtain the theorem [5].

**Lemma (the conullity identity).** If \( T \) is a conullity operator at \( m \), then

\[
\varepsilon_{x,y,z} B_{xy}(T(z)) = 0 \quad \text{for all } x, y, z \in C_k(m).
\]

**Proof.** Let \( T \) be the conullity operator associated with \( u \in N_k(m) \). The second Bianchi identity for \( B \) states that \( \varepsilon_{x,y,z} \nabla_x(B)_{yz}(u) = 0 \). Using the definition of \( B \) in terms of \( A \), and the relation \( \nabla_x(B)_{yz}(u) = \nabla_x(B_{yz})u - B_{yz,x}u - B_{xy,z}u - B_{y,z}x(u) \), where \( X, Y, Z, \) and \( U \) are extensions of \( x, y, z \) and \( u \), with \( U \) a \( k \)-nullity field, we find that

\[
0 = \varepsilon_{x,y,z} B_{xy}(\nabla_x U) = \varepsilon_{x,y,z} B_{xy}(T(x)).
\]

**Remark.** Although this identity is valid for all values of \( \mu \), it is nontrivial only when there are at least three independent conullity vectors. This is the reason for the \( n - \mu \geq 3 \) hypothesis in the theorem.

**Lemma.** If \( \lambda \) is a real eigenvalue of a conullity operator, then \( \lambda \) is zero.

**Proof.** Let \( T \) be the conullity operator at \( m \) associated with \( u \in N_k(m) \). We may assume that \( u \) is a unit vector because \( T \) is linear in \( u \). As in Theorem (3.1) of [5], we calculate the curvature of \( M^n \) along a unit speed geodesic \( \sigma \) starting at \( m \) in the \( u \) direction. The frame field used in the calculation remains valid for this case [3]. If \( P(t) \) is the matrix of \( T_{e(t)} \) relative to the adapted frame field used in this calculation, we obtain a differentiable matrix-valued function \( P \) that satisfies the differential equation \( P' = -P^2 - kI \). Since \( M^n \) is complete, the domain of \( P \) is the entire real line.

Let \( x \) be an eigenvector of \( P(0) \) with the real eigenvalue \( \lambda \). The relation \( P' = -P^2 - kI \) implies that \( x \) is an eigenvector of any derivative of \( P \) at time zero. Using the power series representation of \( P \) given by Picard iteration, we can deduce that \( x \) is an eigenvector of \( P(t) \) for all \( t \). Thus, we may assume that \( P_{ij}(t) = 0 \) for \( j \neq 1 \). If we set \( p(t) = P_{11}(t) \), we find that \( p \) satisfies the equation \( p' = -p^2 - k \).

We can assume that \( k \neq 0 \), as this case is solved in Theorem (3.1) of [5].

Thus, if \( k < 0 \), \( p(t) = \omega(p_0 + \omega \tanh \omega t)/(\omega + p_0 \tanh \omega t) \), where \( \omega = \sqrt{-k} \), and \( p_0 = p(0) \).
If \( k > 0 \), \( p(t) = p_0 + \omega(t) \tan \omega t \), where \( \omega = \sqrt{1+k} \).

In either case, if \( p_0 \neq 0 \), the denominator of \( p \) would vanish for some value of \( t \), and \( p \) would not be differentiable. Thus, \( \lambda = p_0 = 0 \).

To show that each conullity operator \( T \) vanishes, it suffices to show that the eigenvalues of \( T \) are real and that \( T \) can have no multiple eigenspaces with eigenvalue zero. The proofs of these facts are algebraic in nature, and are similar to Theorems (4.2) and (4.6) of [5], which used the conullity identity for \( R \) and \( T \), the symmetries of \( R \), and the fact that the sectional curvatures of conullity planes were nonzero. In this case, we have the conullity identity for \( B \) and \( T \), the fact that \( B \) shares the symmetries of \( R \), and the fact that \( \langle B_{xy}, x, y \rangle \neq 0 \) for all \( x, y \in C_k(m) \).

Remark. It should also be clear that a theorem analogous to Theorem (2*) of [5] holds. That is, if we replace the hypotheses that \( n-p \) is odd, and that the sectional curvatures of conullity planes are unequal to \( k \), by the condition that the tensor \( B \) is positive or negative definite when restricted to pairs of conullity vectors, then the conclusion of the theorem holds. This is again an algebraic consequence of the theorems in [5].

References


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