MONOTONE MAPPING PROPERTIES OF HEREDITARILY INFINITE DIMENSIONAL SPACES

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1. Introduction. Since the discovery of hereditarily infinite-dimensional (HID) spaces by D. W. Henderson [3], questions have naturally arisen about the topological properties of such spaces. A hereditarily infinite-dimensional space is an infinite-dimensional compact metric space each of whose nondegenerate subcontinua is infinite dimensional.

In a previous paper [7], we studied the structure of HID spaces. In this paper, we consider the behavior of HID spaces under monotone mappings. The principal result of this paper is that, given an arbitrary compact metric space $Y$, there is an HID space $X$ and a monotone map $f: X \to Y$. We also show that an arbitrary HID space can be mapped monotonically onto a space of any preassigned dimension, and that, given an HID space $X$ and a positive integer $n$, there is an $n$-dimensional space $Y$ and a monotone map $f: Y \to X$.

R. H. Bing showed in [2] that each nondegenerate monotone image of a pseudo-arc is a pseudo-arc. The results of this paper show that no similar monotone invariance property holds for spaces of dimension greater than 1.

In this paper, all spaces will be compact metric spaces (compacta). We will be dealing with the Hilbert cube, which we regard as being the product of a countably infinite collection of straight line intervals

$$I^\omega = I_1 \times I_2 \times I_3 \times \cdots,$$

where $I_j = [-1/2^j, 1/2^j]$.

By the dimension of a space we will mean the Menger-Urysohn, or small inductive, definition of dimension, or any equivalent definition (see [5 appendix]).

2. Finite-dimensional monotone images of HID spaces. Given an arbitrary HID space $X$, what can we say about a monotone image of $X$? A monotone image of $X$ can have any preassigned finite dimension, as the following proposition shows. We include this proposition for completeness.

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Proposition 1. Let $X$ be a compact metric space with $\dim X \geq n$. Then there is an $n$-dimensional compact metric space $X_n$ and a monotone map $f: X \to X_n$.

Proof. Since $\dim X \geq n$, there is an essential map $g: X \to I^n$ [6, Theorem III. 5]. Let $f_L$ be the Whyburn factorization of $g$ [4, Theorem 3.40] and let $X_n = f(X)$. Since $f_L$ is a uniformly zero-dimensional mapping, $f_L$ does not lower dimension [6, Proposition III. 7(A)]; hence $\dim X_n \leq n$. On the other hand, $f_L$ is an essential map, for otherwise $g$ would not be essential; hence [6, Theorem III. 5], $\dim X_n \geq n$. The proposition is proved.

It should, perhaps, be remarked that the character of the space $X_n$ is not at all clear. For example, there may well be points at which $X_n$ has dimension less than $n$.

3. An HID continuum which maps monotonically onto $I^1$. Let $C$ be the canonical Cantor set in $I^1$, and let $f$ be the Cantor function on $I^1$ [4, p. 131]. Let $J_1, J_2, \ldots$ be the closures of the components of $I^1\setminus C$ in any convenient exhaustive order, and let $p_i = f(J_i)$. We will construct an HID continuum which maps monotonically onto $I^1$ by substituting an HID continuum $X_i$ for $J_i$, the monotone map being that map obtained by sending $X_i$ to $p_i$.

We regard $I^1$ as being embedded in $I^n$ as the first factor.

Let $X$ be an HID continuum in $I^n$ from $(-1/2, 0, 0, \ldots)$ to $(1/2, 0, 0, \ldots)$ (see [7]).

Define a homeomorphism $h_i$ of $I^n$ into $I^1$ by

$$h_i(x_1, x_2, x_3, \ldots) = (b_i - a_i)x_1 + (a_i + b_i)/2, (b_i - a_i)x_2, (b_i - a_i)x_3, \ldots$$

where $[a_i, b_i] = J_i$. Then $h_i$ takes $I^1$ linearly onto $J_i$ and shrinks all other coordinates of $I^n$ linearly and proportionately. Let $X_i = h_i(X)$. Then $X_i$ is an HID continuum joining the end points of $J_i$,

$$X_i \cap C = J_i \cap C = \{a_i, b_i\},$$

and $X_i \cap X_j = \emptyset$ if $i \neq j$.

Let $K = C \cup \bigcup_{i=1}^{n} X_i$. $K$ is the desired HID continuum, as we shall prove in Lemma 1. Figure 1 gives an indication of what the projection of $K$ on $I_1 \times I_2$ might look like. We remark that $K$ has dimension 1 at each inaccessible point of $C$.

Lemma 1. $K$ is a compact HID continuum which can be mapped monotonically onto $I^1$.

Proof. Any infinite sequence of points in any one $X_i$ has a limit
A point in that $X_i$. Since the diameters of the $X_i$'s tend to zero as $i \to \infty$, and each $X_i$ contains a point of $C$, any infinite sequence of points in an infinite number of the $X_i$ has a limit point in $C$. Thus $K$ is compact.

Since each $X_i$ is a continuum containing the end points of $J_i$, $K$ is connected. Any nondegenerate subcontinuum of $K$ must contain a nondegenerate subcontinuum of some $X_i$, and must therefore be infinite dimensional. This shows that $K$ is HID. Finally, the monotone map of $K$ onto $I^n$ is the map mentioned earlier in this section.

If $X$ is chosen to be hereditarily indecomposable, then the $X_i$'s are the smallest possible preimages of points under a monotone map of $K$ onto $I^n$, since the only hereditarily indecomposable subcontinua of $I^n$ are points.

4. An HID continuum which maps monotonically onto $I^n$. We will now construct an HID continuum $K_\omega$ which maps monotonically onto $I^n$. In this construction, the basic building block will be the HID continuum $K$ of §3.

The plan is to obtain a countable collection of HID spaces, each of which lies in $I^n$ and is the cartesian product of a Cantor set with a homeomorphic copy of $K$. These spaces will be constructed so that their union is an HID continuum which maps monotonically onto a Hilbert cube.

We first single out a countable collection $\{H_i\}$ of Hilbert cubes each of which is properly embedded in $I^n$. Let $p_i$ denote the $i$th prime number, and let $H_i$ denote the Hilbert cube given by

$$H_i = I_{p_1} \times I_{p_2} \times I_{p_3} \times \cdots \times I_{p_i} \times \cdots.$$
Let $H = H_1 \times H_2 \times \cdots$; $H$ is the product of a countable collection of Hilbert cubes and is itself a Hilbert cube. We regard $H$ as being a subset of $I^I$ where all factors of $H$ which are not specified are assumed to be $\{0\}$. (This convention will be used in the following discussion without further comment.)

We will also need to single out the Hilbert cube $H_0 = I_2 \times I_3 \times I_5 \times \cdots$ which is a subcube of $H$.

Now let $K_i$ be a copy of $K$ constructed in the Hilbert cube $H_i$, and let $C_i$ be the canonical Cantor set in $I_{pi}$. Let $M_i = C_1 \times C_2 \times C_3 \times \cdots \times C_{i-1} \times K_i \times C_{i+1} \times \cdots$; $M_i$ is just the cartesian product of a Cantor set with $K_i$. Observe that $M_i$ contains the Cantor set $C_0 = C_1 \times C_2 \times C_3 \times \cdots$ since $C_i \subseteq K_i$.

We define $K_{\infty}$ to be $\bigcup_{i=1}^{\infty} M_i$.

**Theorem 1.** $K_{\infty}$ is a compact, hereditarily infinite-dimensional continuum which can be mapped monotonically onto the Hilbert cube.

**Proof.** (1) $K_{\infty}$ is compact: $K_i$ is compact by Lemma 1, hence $M_i$ is compact, being the product of compact spaces. Any finite union of the $M_i$ is compact, since each $M_i$ is. Let $\{x_j\}$ be a sequence of points with each $x_j$ belonging to a different $M_i$. Since the diameter of $H_i$ is less than $2^{2^{-pi}}$, it follows that the diameter of $M_i$ is less than $2^{2^{-pi}}$. Thus if $x_j \in M_i$, it follows that there is a point $y_j \in C_0$ such that $d(x_j, y_j) < 2^{2^{-pi}}$. Since $C_0$ is compact, the sequence $\{y_j\}$ has a limit point $y_0 \in C_0$; and it follows that some subsequence of the sequence $\{x_j\}$ also converges to $y_0$. But by construction, $C_0 \subseteq K_{\infty}$; hence $K_{\infty}$ is compact.

(2) $K_{\infty}$ is connected: Let $x, y \in K_{\infty}$. We will construct a continuum in $K_{\infty}$ containing both $x$ and $y$. We may assume without loss of generality that both $x$ and $y$ belong to $C_0$.

Let $x = (x_1, x_2, \cdots), x_i \in C_i \subseteq H_i$, $y = (y_1, y_2, \cdots), y_i \in C_i \subseteq H_i$. In $M_i$, there is a continuum containing both $(y_1, y_2, \cdots, y_{i-1}, x_i, x_{i+1}, \cdots)$ and $(y_1, y_2, \cdots, y_{i-1}, y_i, x_{i+1}, \cdots)$; namely,

$$y_1 \times y_2 \times \cdots \times y_{i-1} \times K_i \times x_{i+1} \times \cdots.$$ 

Call this $L_i$. Let $L = \bigcup_{i=1}^{\infty} L_i$. Then $L$ is a compact set which contains both $x$ and $y$; $L$ will be shown to be connected when we show that $L_i \cap L_{i+1} \neq \emptyset$ for all $i$. But by construction we have

$$(y_1, y_2, \cdots, y_{i-1}, y_i, x_{i+1}, \cdots) \in L_i$$

and

$$(y_1, y_2, \cdots, y_{i+1}, x_{i+1}, \cdots) \in L_{i+1}.$$
Connectedness of \( L \) implies connectedness of \( K_\infty \).

(3) \( K_\infty \) is hereditarily infinite dimensional: Observe first that \( C_0 \) is 0-dimensional.

Let \( Y \) be a nondegenerate subcontinuum of \( K_\infty \), and let \( x \in Y \setminus C_0 \). Then \( x \in M_j \) for some integer \( j \), and \( x_j \), the \( j \)-th coordinate of \( x \), does not belong to \( C_j \). Then there is a neighborhood \( U \) of \( x \) such that for any \( y \in U \), \( y \not\in C_j \). Now if \( p \in M_i \), \( i \neq j \), it follows from the construction that \( p_j \in C_j \). Thus \( x \) is not a limit point of \( U \cap Y \).

This implies that \( x \) belongs to a nondegenerate subcontinuum \( Y' \) of \( M_i \), and \( Y' \) must lie in a copy of \( K_i \). Then \( Y' \) is infinite dimensional; hence \( Y \) is also infinite dimensional.

(4) There is a monotone map of \( K_\infty \) onto \( H_0 \): Let \( \phi_n \) be the Cantor map on \( I_n \). Define \( f: K_\infty \rightarrow H_0 \) by

\[
f(x) = (\phi(x_2), 0, 0, \ldots, \phi(x_3), 0, 0, \ldots, \phi(x_4), 0, 0, \ldots).
\]

\( f \) is onto, since \( C_0 \subseteq K_\infty \) and \( f: C_0 \rightarrow H_0 \) is onto. In fact, for \( y \in H_0 \), \( f^{-1}(y) \) is the intersection of \( K_\infty \) with the Hilbert cube

\[
\phi^{-1}(y_2) \times I_2 \times I_3 \times \cdots \times \phi^{-1}(y_3) \times I_2 \times \cdots.
\]

To show that \( f \) is monotone, we must exhibit in \( f^{-1}(y) \) a continuum containing any two preassigned points of \( f^{-1}(y) \). This proof is entirely analogous to the proof that \( K_\infty \) is connected (part (2) of this theorem) and we therefore omit it. Since \( H_0 \) is homeomorphic with \( I^n \), the proof is complete.

**Corollary.** For each \( n \), there is a monotone map of \( K_\infty \) onto an \( n \)-cell.

**Proof.** Follow \( f \) by the projection of \( I^n \) onto its first \( n \) factors. We remark that this corollary is an immediate consequence of Theorem 2; however the proof we give here is somewhat neater in this special case.

5. **HID compacta as monotone preimages of arbitrary compacta.**

We saw in the previous section that there is an HID space which maps monotonically onto \( I^n \). The question naturally arises whether, given any compact metric space, there is an HID space which maps monotonically onto it. That the answer is yes is a corollary of Theorem 1, but Theorem 2 gives us a stronger result. We first need the following lemma, which is an extension of Theorem 1:
Lemma 2. There is an HID space $L$ and a monotone map $g: L \to I^w$ such that for each $p \in I^w$, $g^{-1}(p)$ is HID.

Proof. Let $K_w$ be the HID space of Theorem 1, and regard $K_w$ as being embedded in $I^w$. Let $C_0 \subseteq K_w$ be the Cantor set described in the previous section. Let $I^w'$ be another Hilbert cube, and let $X$ be a hereditarily indecomposable HID continuum in $I^w'$ which contains the point $(0, 0, 0, \ldots)$. Then $L$ is the subset of $I^w \times I^w'$ given by $L = (K_w \times (0, 0, 0, \ldots)) \cup (C_0 \times X)$. Intuitively, $L$ is obtained by tacking a copy of $X$ onto $K_w$ at each point of $C_0$. If $\pi$ is the projection map of $I^w \times I^w'$ onto $I^w$, and if $f$ is the monotone map of $K_w$ onto $I^w$, then the map $g = f|_L \pi$ is a monotone map of $L$ onto $I^w$, and it is clear that for each $p \in I^w$, $g^{-1}(p)$ is HID. This completes the proof of the lemma.

Theorem 2. Let $X$ be a compact metric space. Then there is an HID compactum $Y$ and a monotone map $f: Y \to X$ such that for each $p \in X$, $f^{-1}(p)$ is HID. Moreover, the components of $X$ are in 1-1 correspondence with the components of $Y$ under the correspondence $c(y) \leftrightarrow c(f(y))$.

Proof. Let $X$ be embedded in $I^w$, and let $g$ be the monotone map given in Lemma 2. Let $Y = g^{-1}(X)$. Since $L$ is compact and $g$ is a monotone map, the preimage of each component of $X$ is a component of $Y$. $Y$ is HID since it is an infinite dimensional subspace of an HID space. This completes the proof of the theorem.

Corollary. If $X$ is a separable metric space, all conclusions of Theorem 2 hold except perhaps compactness of $Y$.

Proof. $X$ can be embedded in a compactum $\hat{X}$ of the same dimension [6, Theorem V. 6]. Apply Theorem 2 to $\hat{X}$.

6. HID compacta as monotone images of finite dimensional compacta. In §2, we saw that any HID space has a monotone image of any prescribed dimension.

We now observe that any HID space is the monotone image of some space of any prescribed dimension. This is an almost immediate corollary of a result announced by R. D. Anderson in [1], which states that any compact locally connected metric continuum is a monotone-open image of the universal curve under a map $f$ and, moreover, $f$ can be chosen so that each point preimage is homeomorphic to the universal curve. In particular, this is true of the Hilbert cube $I^w$; and if $X$ is any compact metric space, which we regard as being embedded in $I^w$, then $f|_{f^{-1}(X)}$ is a monotone-open map of a subset of the universal curve onto $X$. If we let $Y = f^{-1}(X)$,
then \( Y \times I^{n-1} \) is an \( n \)-dimensional space and \( f \times \{ \text{projection onto origin} \} \) is a monotone map of \( Y \times I^{n-1} \rightarrow X \). Moreover, it follows that the preimage of every point under this map has dimension \( n \). For emphasis, we summarize this discussion as follows:

**Proposition 2.** Let \( X \) be a compact metric space, and let \( n \) be any positive integer. Then there is a compact metric space \( Y \) of dimension \( n \), and a monotone map \( f: Y \rightarrow X \). Moreover, \( f^{-1}(x) \) has dimension \( n \) for each \( x \in X \).

In the case where \( X \) is HID, the space \( Y \) fails to be locally connected since local connectedness is preserved by monotone mappings and local connectedness together with our other hypotheses would imply arcwise connectedness of \( X \). It would be of interest to have a definitive description of \( Y \) in this case; such a description might help in understanding the structure of HID spaces.

It might be remarked that Proposition 2 can also be obtained by modifying the constructions in §§3, 4, and 5 to use \( n \)-cells instead of HID continua; however, there is little point in doing that since the proof given here is neater and more intuitively appealing.

7. Questions.
(a) Is every hereditarily indecomposable continuum the monotone image of a hereditarily indecomposable HID continuum?
(b) Is there an HID continuum \( K' \) and a monotone map \( f: K' \rightarrow I \) such that each point preimage is a hereditarily indecomposable HID continuum?

**References**


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