PROOF OF ANDREWS' CONJECTURE ON
PARTITION IDENTITIES

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1. Introduction. As in [2], we denote by $B_d(n)$ the number of
partitions of $n$ of the form $n=b_1+b_2+\cdots+b_s$, satisfying the
following conditions:

(i) $b_i-b_{i+1}\geq d$,

(ii) if $d|b_i$, then $b_i-b_{i+1}>d$.

We denote by $C_d(n)$ the number of partitions of $n$ satisfying the above
conditions and additionally

(iii) $b_s>d$.

These two partition functions appear in several of the well-known
identities in the theory of partitions. Thus, the first of the Rogers-
Ramanujan identities [5, p. 291] states that $B_1(n)$ is equal to the
number of partitions of $n$ into parts $\equiv \pm 1 \pmod{5}$, while the second
Rogers-Ramanujan identity asserts that $C_1(n)$ is equal to the number
of partitions of $n$ into parts $\equiv \pm 2 \pmod{5}$. H. Göllnitz [3] and B.
Gordon [4] proved independently that $B_2(n)$ is equal to the number
of partitions of $n$ into parts $\equiv 1, 4, 7 \pmod{8}$, while $C_2(n)$ is the
number of partitions of $n$ into parts $\equiv 3, 4, 5 \pmod{8}$. I. J. Schur [6]
has proved that $B_3(n)$ is equal to the number of partitions of $n$ into parts
$\equiv \pm 1 \pmod{6}$. For $d>3$, a theorem by the author [1, p. 713]
can easily be extended to prove that $B_d(n)$ is not equal to the number
of partitions of $n$ into parts taken from any set of integers whatsoever.

Andrews [2, p. 441] has proved a certain identity involving $C_3(n)$,
but states that he has not been able to obtain any simple partition-
theoretic interpretation of this identity. He conjectures "that
Alder's result for $B_d(n)$ is also valid for $C_d(n)$ with $d>2"."

It is the object of this paper to prove this conjecture of Andrews.
In fact, we prove a more general result, namely that this conjecture
is valid if we replace (iii) above by

(iv) $b_s\geq m$, 

where $m\geq 2^1$, so that Andrews' conjecture is the special case where
$m=d+1$. This result is stated in the following

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1 The author is grateful to the referee for suggesting this even more general in-
equality than that contained in the author's original manuscript and for proposing a
 corresponding simplification of the proof.
**Theorem.** The number $C_{d,m}(n)$ of partitions of $n$ into parts of the form $n = b_1 + b_2 + \cdots + b_s$, with $b_i - b_{i+1} \geq d$, and if $d \mid b_i$, then $b_i - b_{i+1} > d$, and $b_s \leq m$, where $m \geq 2$, is not equal to the number of partitions of $n$ into parts taken from any set of integers whatsoever if $d > 2$.

As a special case of this theorem it follows that there cannot exist a dual to Schur's Theorem in the sense that the second of the Rogers-Ramanujan identities is a dual to the first one and it also explains why Andrews has not been able to obtain any simple partition-theoretic interpretation of his Theorem 3 in [2].

2. **Proof of the theorem.** We suppose that the theorem is false and that there exists such a set of integers $a_1 < a_2 < a_3 < \cdots$; denote this set by $A$ and the number of partitions of $n$ into parts taken from this set by $p_A(n)$. Let $n$ be any integer for which $C_{d,m}(n) \geq 2$, then $n \geq (m+d) + m = 2m + d$. Hence $C_{d,m}(n) = 1$ for $m \leq n < 2m + d$; in particular, $C_{d,m}(2m+2) = 1$. Now, since $C_{d,m}(n) = 0$ for $1 \leq n < m$, it follows that for $m \geq 3$, $a_1 = m$, $a_2 = m + 1$, $a_3 = m + 2$. But then $p_A(2m+2) \geq 2$, since $2m+2 = m + (m+2) = (m+1) + (m+1)$, which is a contradiction. If $m = 2$, then $a_1 = 2$, $a_2 = 3$, and hence $p_A(6) \geq 2$, which is again a contradiction.

**References**


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