PROOF OF ANDREWS’ CONJECTURE ON PARTITION IDENTITIES

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1. Introduction. As in [2], we denote by $B_d(n)$ the number of partitions of $n$ of the form $n = b_1 + b_2 + \cdots + b_r$, satisfying the following conditions:
   (i) $b_i - b_{i+1} \geq d$,
   (ii) if $d | b_i$, then $b_i - b_{i+1} > d$.
We denote by $C_d(n)$ the number of partitions of $n$ satisfying the above conditions and additionally
   (iii) $b_s > d$.

These two partition functions appear in several of the well-known identities in the theory of partitions. Thus, the first of the Rogers-Ramanujan identities [5, p. 291] states that $B_1(n)$ is equal to the number of partitions of $n$ into parts $\equiv \pm 1 \pmod{5}$, while the second Rogers-Ramanujan identity asserts that $C_1(n)$ is equal to the number of partitions of $n$ into parts $\equiv \pm 2 \pmod{5}$. H. Göllnitz [3] and B. Gordon [4] proved independently that $B_2(n)$ is equal to the number of partitions of $n$ into parts $\equiv 1, 4, 7 \pmod{8}$, while $C_2(n)$ is the number of partitions of $n$ into parts $\equiv 3, 4, 5 \pmod{8}$. I. J. Schur [6] has proved that $B_3(n)$ is equal to the number of partitions of $n$ into parts $\equiv \pm 1 \pmod{6}$. For $d > 3$, a theorem by the author [1, p. 713] can easily be extended to prove that $B_d(n)$ is not equal to the number of partitions of $n$ into parts taken from any set of integers whatsoever.

Andrews [2, p. 441] has proved a certain identity involving $C_3(n)$, but states that he has not been able to obtain any simple partition-theoretic interpretation of this identity. He conjectures “that Alder's result for $B_d(n)$ is also valid for $C_d(n)$ with $d > 2$.”

It is the object of this paper to prove this conjecture of Andrews. In fact, we prove a more general result, namely that this conjecture is valid if we replace (iii) above by
   (iv) $b_s \geq m$,
where $m \geq 2^1$, so that Andrews’ conjecture is the special case where $m = d + 1$. This result is stated in the following

1 The author is grateful to the referee for suggesting this even more general inequality than that contained in the author’s original manuscript and for proposing a corresponding simplification of the proof.
Theorem. The number $C_{d,m}(n)$ of partitions of $n$ into parts of the form $n = b_1 + b_2 + \cdots + b_s$, with $b_i - b_{i+1} \geq d$, and if $d | b_i$, then $b_i - b_{i+1} > d$, and $b_s \geq m$, where $m \geq 2$, is not equal to the number of partitions of $n$ into parts taken from any set of integers whatsoever if $d > 2$.

As a special case of this theorem it follows that there cannot exist a dual to Schur's Theorem in the sense that the second of the Rogers-Ramanujan identities is a dual to the first one and it also explains why Andrews has not been able to obtain any simple partition-theoretic interpretation of his Theorem 3 in [2].

2. Proof of the theorem. We suppose that the theorem is false and that there exists such a set of integers $a_1 < a_2 < a_3 < \cdots$; denote this set by $A$ and the number of partitions of $n$ into parts taken from this set by $p_A(n)$. Let $n$ be any integer for which $C_{d,m}(n) \geq 2$, then $n \geq (m+d)+m = 2m+d$. Hence $C_{d,m}(n) = 1$ for $m \leq n < 2m+d$; in particular, $C_{d,m}(2m+2) = 1$. Now, since $C_{d,m}(n) = 0$ for $1 \leq n < m$, it follows that for $m \geq 3$, $a_1 = m$, $a_2 = m+1$, $a_3 = m+2$. But then $p_A(2m+2) \geq 2$, since $2m+2 = m + (m+2) = (m+1) + (m+1)$, which is a contradiction. If $m = 2$, then $a_1 = 2$, $a_2 = 3$, and hence $p_A(6) \geq 2$, which is again a contradiction.

References


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