COMMUTATIVE SEMIGROUP LAWS

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1. Introduction. It is a consequence of B. H. Neumann's classification of group identities [3, Theorem 19.1, p. 523], that the lattice of Abelian group varieties is distributive. The lattice of varieties of algebras in one unary operation is also distributive [2], but the lattice of commutative semigroup varieties is not modular [4]. Here we discuss a distributive sublattice of this nonmodular lattice.

By variety we will mean commutative semigroup variety. (Lemmas 1 and 2, however, are true for semigroup varieties, not necessarily commutative.) The semigroups need not have a unit-element. We will be mainly concerned with laws of the form \( s = sxa \) where \( s \) is a term (word in the variables), \( x \) is variable and \( a \) is a positive integer. We call such a law an \( L \)-law and call a variety which can be defined by a set \( \{ s_i = s_i x^{a_i} \} \) of \( L \)-laws, an \( L \)-variety.

2. \( L \)-Laws. Exponents of variables will always be positive integers. Lemma 1 is easily proved by induction on \( k \), where \( b = ka \).

**Lemma 1.** Let \( s \) be a term and \( x \) be a variable. If \( b \) is a multiple of \( a \), then \( s = sxb \) holds in the variety defined by \( s = sxa \).

**Lemma 2.** Let \( s, t \) be terms and \( x \) be a variable. If \( d \) is the greatest common divisor of \( a \) and \( b \), then \( s = sxd \) holds in the variety defined by \( s = sxa \) and \( t = txb \).

**Proof.** The substitution of \( x \) for each variable in \( t = txb \) yields \( x^p = x^{p+b} \), where \( p \) is some positive integer. Hence we have \( sx^p = sx^{p+b} \), and thus, by Lemma 1, \( sx^p = sx^{p+jb} \), \( j = 1, 2, \ldots \).

From \( s = sx^a \) we obtain \( sx^p = sx^{p+ia} \), \( i = 1, 2, \ldots \). Hence \( sx^p = sx^{p+ia+jb} \), \( i, j = 0, 1, 2, \ldots \).

Thus, by an elementary property of nonnegative integers, \( sx^p = sx^{p+ia+jb} \) for some nonnegative integers \( k \).

From this last law and \( s = sx^a \), we obtain \( sx^p = sx^{p+d} \). So \( sx'^a = sx'^ax^d \), where \( ra \geq p \). Hence, since \( s = sx'^a \), we have \( s = sx^a \).

By \( n(x, s) \) we mean the number (\( \geq 0 \)) of occurrences of a variable \( x \) in a term \( s \).

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Lemma 3. If \( n(x, s) \geq n(y, s) \), then \( s = sx^a \) holds in the variety defined by \( s = sy^a \).

Proof. Suppose \((i, j)\), where \(i, j\) are positive integers, denotes the term \(x^iy^j\). Then each of the following laws (after the first) can be obtained from the previous one.

\[
\begin{align*}
(p, q) &= (p, q + a), \\
(q + ka, q) &= (q + ka, q + a), & \text{if } q + ka \geq p, \\
(q + ka, q) &= (q + ka, q + ka), \\
(q + ka, q) &= (q, q + ka), \\
(p + ka, q) &= (p, q + ka), & \text{if } p \geq q.
\end{align*}
\]

Similarly, from \( s = sy^a \) we can derive \( sx^{ka} = sy^{ka} \), where \(k\) is a positive integer such that \( n(y, s) + ka \geq n(x, s) \).

By a familiar argument, the last law, with \( s = sy^a \), yields \( s = sx^{ka} \).

From \( s = sy^a \), by substitution, we have \( t = tx^a \) for some term \( t \). Hence, since \( a \) is the g.c.d. of \( ka \) and \( a \), we have, by Lemma 2, \( s = sx^a \).

We define a simple closure operation on the set of terms as follows:

(i) The closure \( \text{cl}(S) \) of a set \( S \) of terms is the union of the closures of the one element subsets of \( S \).

(ii) For terms \( s, t, t \in \text{cl}(\{s\}) = \text{cl}(s) \), in case there is a function from the set of variables of \( s \) into the set of variables of \( t \) such that if distinct variables \( x_1, \ldots, x_m \) are mapped to a variable \( x \), then

\[
n(x_1, s) + \cdots + n(x_m, s) \leq n(x, t).
\]

Thus for variables \( x, y \) and any term \( t, x^iy^j, x^it \) are in \( \text{cl}(x^iy^j) \).

For any term \( t \) denote by \( Sg(t) \) the free commutative semigroup generated by the variables occurring in \( t \). Then (ii) says: There is a homomorphism \( \phi: Sg(s) \rightarrow Sg(t) \), that maps variables into variables, such that \( t \) lies in the ideal generated by \( \phi(s) \).

Lemma 4. If \( n(x, t) \geq n(x, s_1) \) and \( t \in \text{cl}(s_2) \) then \( t = tx^a \) holds in the variety defined by the laws \( s_1 = s_1x^a \), \( s_2 = s_2x^a \).

Proof. It is readily seen that from \( s_1 = s_1x^a \) we can obtain a law \( s = sy^a \), where \( y \) is a variable not in \( t, s \) is a term that does not contain \( x \) or any of the variables of \( t \), and \( n(x, s_1) = n(y, s) \). Then from \( s = sy^a \), we have \( ts = tsy^a \). Also

\[
n(x, ts) = n(x, t) \geq n(x, s_1) = n(y, s) = n(y, ts).
\]

Thus by Lemma 3, we have \( ts = ttx^a \).

Since \( t \in \text{cl}(s_2) \), we find, using \( s_2 = s_2x^a \), that \( t = tu^a \) for some variable \( u \). The substitution of \( u^a \) for each variable of \( s \) in \( ts = ttx^a \) leads to
for some positive integer $k$. This last law, with $t = tu^a$, yields $t = tx^a$.

**Lemma 5.** Let $E = \{s_i = s_i x^{a_i}\}$ be a set of $L$-laws. Then the $L$-law $t = tx^b$ holds in the variety defined by $E$ if and only if

(i) $b$ is a multiple of $\text{g.c.d.} \{a_i\}$,
(ii) $n(x, t) \geq \min \{n(x, s_i)\}$ and,
(iii) $t \in \text{cl} (\{s_i\})$.

**Proof.** Let $\tau$ denote the law $t = tx^a$. Suppose $\tau$ holds in the variety $V$ defined by $E$. The cyclic group of order $\text{g.c.d.} \{a_i\}$ is (as a semigroup) an algebra of $V$, hence $\tau$ must satisfy condition (i).

Let $p = \min \{n(x, s_i)\}$ and suppose $p > 0$ (otherwise (ii) is trivial). Let $A$ be the commutative semigroup with two generators $a, b$ defined by $a^p = a^{p+1}, b^p = b^{p+1}$. Clearly $A$ is in $V$ and if $n(x, t) < p$, $\tau$ cannot hold in $A$. (In $\tau$ substitute $a$ for $x$ and $b$ for the other variables, if any, of $\tau$.)

Define an equivalence relation $R$ on the set of terms by $s \equiv t$ if and only if $s = t$ holds in every commutative semigroup. Let $[s]$ denote the equivalence class containing $s$ and define a binary operation on the set $Q$ of equivalence classes by $[s][t] = [st]$. Let $P$ be the set of all $[s]$ such that $s \in \text{cl} (\{s_i\})$. Then $P$ is an ideal of $Q$. Let $B = Q/P$ be the Rees factor semigroup of $Q$ modulo $P$ [1, p. 17]. Then $B$ is in $V$ and for $\tau$ to hold in $B$, $\tau$ must satisfy condition (iii).

Conversely suppose conditions (i)–(iii) are satisfied. Then by Lemmas 1, 2, and 4, $\tau$ holds in the variety defined by $E$.

3. The lattice of $L$-varieties. It follows from Lemma 5 that if two sets $\{s_i = s_i x^{a_i}\}$ and $\{t_i = t_i x^{b_i}\}$ of $L$-laws define the same variety then

(i) $\text{g.c.d.} \{a_i\} = \text{g.c.d.} \{b_i\}$,
(ii) $\min \{n(x, s_i)\} = \min \{n(x, t_i)\}$, and
(iii) $\text{cl} (\{s_i\}) = \text{cl} (\{t_i\})$.

Thus if $V$ is the variety defined by a set $\{s_i = s_i x^{a_i}\}$ we let

(i) period $V = \text{g.c.d.} \{a_i\}$,
(ii) level $V = \min \{n(x, s_i)\}$, and
(iii) scope $V = \text{cl} (\{s_i\})$.

Let $\phi$ be a law $s = t$ with $a = n(x, s)$ and $b = n(x, t)$. We make the following definitions

(i) Period of $x$ in $\phi = |a - b|$.
   Period $\phi = \text{g.c.d.}$ of the periods of the variables of $\phi$.
(ii) Level of $x$ in $\phi = \min (a, b)$.
   Level of $\phi = \text{minimum}$ of the levels of the variables of $\phi$ with nonzero periods.
(iii) Scope $\phi = \text{cl}({s, t})$.
(Thus the period, level and scope respectively of $s = sx^a$ is $a$, $n(x, s)$, and $\text{cl}(s)$.)

We say a law $\phi$ is trivial in case $\phi$ holds in every commutative semigroup.

**Theorem 1.** Let $V$ be an L-variety. A nontrivial law $\phi$ holds in $V$ if and only if

(i) period $\phi$ is a multiple of period $V$
(ii) level $\phi \geq$ level $V$ and
(iii) scope $\phi \subseteq$ scope $V$.

**Proof.** We omit the proof of the necessity, since it is similar to the proof of the necessity in Lemma 5.

Suppose $\phi: s = t$ satisfies conditions (i), (ii), and (iii). Let $x_1, \ldots, x_m$ ($y_1, \ldots, y_n$) be the variables that appear more often in $s(t)$ than in $t(s)$ and let $c_i(d_i)$ be the period in $s = t$ of $x_i(y_i)$. By Lemma 5 we have that $s = sy_1^{d_1}, \ldots, s = sy_n^{d_n}, t = tx_1^{d_1}, \ldots, t = tx_m^{d_m}$ hold in $V$. Hence $s = sy_1^{d_1} \cdots y_n^{d_n}$ and $t = tx_1^{d_1} \cdots x_m^{d_m}$ hold in $V$. From these two laws and the trivial law $sy_1^{d_1} \cdots y_n^{d_n} = tx_1^{d_1} \cdots x_m^{d_m}$, we have that $\phi$ holds in $V$.

**Lemma 6.** Given two L-varieties with levels $p$, $q$ and scope $A$, $B$ respectively there is an L-variety with level $\max(p, q)$, scope $(A \cap B)$, and period $a$ for any $a > 0$.

**Proof.** Let $C = \{t \in A \cap B | n(x, t) \geq \max(p, q)\}, E = \{t = tx^a | t \in C\}$, then the variety defined by $E$ has the desired properties.

We denote the join of two varieties $V$, $W$ by $V + W$. The next theorem follows from Theorem 1, Lemma 6, and an observation dual to Lemma 6, involving $\min$ instead of $\max$ and $\cup$ instead of $\cap$.

**Theorem 2.** Let $V$ and $W$ be L-varieties. Then $V \cap W$ and $V + W$ are L-varieties and the mapping

$V \rightarrow (\text{period } V, \text{level } V, \text{scope } V)$

between the lattice of L-varieties under $\cap$ and $+$ and the direct product of

(i) the lattice of positive integers under g.c.d. and l.c.m.,
(ii) the lattice of nonnegative integers under $\min$ and $\max$, and
(iii) the dual of the lattice of all closed sets of terms under $\cap$ and $\cup$

is an injective homomorphism.

The next theorem follows immediately from Theorem 2.

**Theorem 3.** The L-varieties form a distributive sublattice of the lattice of commutative semigroup varieties.
The mapping considered in Theorem 2 is not an isomorphism. For example, there is no $L$-variety with scope equal to the set of all terms and level $\geq 2$.

4. On other laws. Since the lattice of varieties is not modular, not every variety is an $L$-variety. A simple example of a non-$L$-variety is the variety defined by $xy^2 = x^2y$; other examples occur in [4]. We consider two types of laws that define $L$-varieties.

**Theorem 4.** Let $s, t$, be terms. The variety defined by $s = st$ is an $L$-variety.

**Proof.** We show $\phi: s = st$ is equivalent to the $L$-law $\sigma: s = sx^n$ where $\alpha = \text{period } \phi$ and $n(x, s) = \text{level } \phi$. (Thus $x$ is a variable of $t$ such that $n(x, s) = \text{level } \phi$.) By Theorem 1 from $\alpha$, we have $\phi$.

Conversely the substitution of $x$ for each variable of $\phi$ and the substitution of $x^p$ for $x$ and $x$ for the other variables, if any, of $\phi$ yields laws in $x$ of periods $q+b$ and $q+2b$ respectively, where $q$ is some nonnegative integer and $b$ is the period of $x$ in $\phi$. Thus, since $\text{g.c.d.} (q+b, q+2b)$ divides $b$, we have $x^p = x^{p+b}$, where $p$ is some positive integer. Similarly we can obtain $L$-laws of the periods of the other variables of $\phi$ in $t$, so we have an $L$-law of period $\phi = \text{period } \sigma$.

From $s = st$ and $x^p = x^{p+b}$ respectively, we have $s = stk$ and $x^{ka} = x^{2ka}$, where $ka \geq p$. These last two laws lead to $s = sx^{ka}$, an $L$-law of level $\sigma$ and scope $\sigma$. Thus we have $\sigma$.

**Theorem 5.** If some variable occurs in only one of the terms, $s, t$, then the variety defined by $s = t$ is an $L$-variety.

**Proof.** Suppose $n(x, s) = 0$ and $n(x, t) = p > 0$. By the previous theorem, it suffices to show that $s = t$ is equivalent to the two laws $s = sp^t$, $t = sp^t$. It is obvious that these laws imply $s = t$. On the other hand the substitution of $sx$ for $x$ in $s = t$ leads to $s = sp^t$, which with $s = t$, yields $t = sp^t$.

**References**


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