A COUNTABLE MINIMAL URYSOHN SPACE IS COMPACT

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1. Introduction. A topological space $X$ is said to be a Urysohn space provided that every pair of distinct points of $X$ have disjoint closed neighborhoods. In [3] C. T. Scarborough and A. H. Stone prove that a countable minimal regular space is compact, and in [4] Scarborough asks if there exist noncompact minimal Urysohn spaces which are countable.

The principal result of this note is the following:

**Theorem.** A countable minimal Urysohn space is compact.

The terminology used here coincides with that in [3] and [4]. We shall denote the set of positive integers by $N$.

2. The theorem. According to [4, Theorem 14], a countable U-closed space has an isolated point. Our first lemma strengthens this result.

**Lemma 1.** Let $X$ be a U-closed space, let $I$ be its set of isolated points, and suppose that $X - I$ is countable. Then $\overline{I} = X$.

**Proof.** Suppose that there exists a nonempty open set $V$ such that $V \cap I = \emptyset$. Let $X - I = \{x_n | n \in N\}$. An inductive argument shows that there exists a descending sequence $V_n$, $n \in N$, of nonempty open subsets of $V$ such that for every $k \in N$ there exists a neighborhood $W_k$ of $x_k$ whose closure misses the closure of $V_k$. Then the filter base $\{V_n | n \in N\}$ is a $U$-filter on $X$ which has void adherence.

**Lemma 2.** Let $X$ be a U-closed space, $I$ its set of isolated points, and $\mathcal{F}$ a countable filter base on $I$. Then $\mathcal{F}$ has an adherent point.

**Proof.** Let $\mathcal{F} = \{F_n | n \in N\}$. For each $n$ choose a point $x_n \in \bigcap \{F_i | i \leq n\}$ and let $C = \{x_n | n \in N\}$. If there is a point $x \in \overline{C} - C$, then $x$ is an adherent point of $\mathcal{F}$. If $\overline{C} = C$ and $\cap \mathcal{F} = \emptyset$, then the open- and-closed sets $\{x_j | j \geq n\}$, $n \in N$, generate a $U$-filter on $X$ which has void adherence.

**Proof of the Theorem.** Let $X$ be a countable minimal Urysohn space.

In [4] Scarborough proves that a minimal Urysohn space is semi-regular. In [2] Katětov proves that a semiregular absolutely closed space is minimal Hausdorff and that a Urysohn minimal Hausdorff

Received by the editors September 20, 1968.
space is compact (this latter result is also obtained in [1]). Thus it suffices for us to prove that $X$ is absolutely closed.

Since $X$ is a Lindelöf space, an easy argument shows that if every countable open filter base on $X$ has an adherent point, then $X$ is absolutely closed.

Let $\mathcal{F}$ be a countable open filter base on $X$, and let $I$ be the set of isolated points of $X$. Then $\mathcal{F} | I$ is a filter base by Lemma 1, so $\mathcal{F} | I$ and, hence, $\mathcal{F}$ have nonvoid adherence by Lemma 2.

**Corollary.** A countable $U$-closed space is absolutely closed.

**References**