1. Introduction. In this paper, we describe methods of imbedding a Hausdorff space $X$ in a compact space $\overline{X}$ so that each function in a given family of continuous functions on $X$ has a continuous extension to $\overline{X}$ and the family of extensions separates the points of $\overline{X} - X$. In particular, if $X$ is completely regular but not locally compact, then we shall exhibit a non-Hausdorff compactification which contains $X$ as an open subset and is bigger than the Stone-Čech compactification of $X$. (Of course, every compactification of $X$ is non-Hausdorff if $X$ is not completely regular.) We shall also show that the completion of a metric space $M$ may be obtained as a subset of a compactification of $M$ by a rather simple construction.

By a compactification of a Hausdorff space $X$, we mean a compact space $\overline{X}$ which contains, as a dense subset, the image of $X$ under a fixed homeomorphism $f$. We usually do not distinguish between $X$ and $f(X)$, and we say that $\overline{X}$ contains $X$ as a dense subset. In what follows, $X$ is always a noncompact Hausdorff space, $\Delta \overline{X}$ denotes the closure of $\overline{X} - X$ in $\overline{X}$, and a mapping is always a continuous function. If $\overline{X}$ is Hausdorff, we say that $\overline{X}$ is a Hausdorff compactification of $X$. If $\overline{X}$ is not Hausdorff, however, we still assume that it satisfies the following properties:

I. Compact subsets of $X$ are closed in $\overline{X}$.

II. Any two distinct points $x$ and $y$ in $\Delta \overline{X}$ can be separated by disjoint open sets; i.e., there exist open sets $U$ and $V$ in $\overline{X}$ with $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

III. For each point $x \in X$ there is at most one point $z \in \overline{X} - X$ such that $x$ and $z$ cannot be separated by disjoint open sets in $\overline{X}$.

Clearly, II and III are necessary conditions for the points in $\Delta \overline{X}$ to be separated by a family of continuous functions from $\overline{X}$ into a Hausdorff space; we shall show later that, together with Condition I, they are also sufficient. The following properties of $\overline{X}$ are consequences of I, II, and the fact that $X$ is dense in $\overline{X}$.  

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Proposition 1.1. If $\overline{X}$ is a compactification of $X$, then:

(i) Two distinct points in $X$ can be separated by disjoint open sets in $\overline{X}$.

(ii) The space $\overline{X}$ is $T_1$.

(iii) In $\overline{X}$, a sequence converges to at most one point.

(iv) If $X$ satisfies the first axiom of countability then compact subsets of $\overline{X}$ are closed.

(v) If the compactification $\overline{X}$ satisfies the first axiom of countability, then $\overline{X}$ is Hausdorff.

(vi) $X$ is locally compact if and only if $X$ is open in $\overline{X}$ and $\overline{X}$ is Hausdorff.

Example. Following Arens [1], let $X$ be the set of all pairs of nonnegative integers such that each point other than $(0, 0)$ is an open set and every neighborhood of $(0, 0)$ contains all but a finite number of points in all but a finite number of columns $C_n$, where $C_n = \{(m, m) : m \in \mathbb{Z}^+\}$. Then $X$ is not first countable. Let $\overline{X}$ be the space $X$ together with a point $P$ whose neighborhoods omit at most a finite number of points in $X$. Any nonrepeating sequence with an infinite number of points in each column $C_n$ converges to $P$ and has cluster point $(0, 0)$. Moreover, $\overline{X} - \{(0, 0)\}$ is compact but not closed in $\overline{X}$.

Now let $\overline{X}$ and $\bar{X}$ be two compactifications of $X$. By the notation $\overline{X} \cong \bar{X}$ we mean there is a mapping $T$ of $\overline{X}$ onto $\bar{X}$ such that $T|X$ is the identity map. (To be more accurate, we should say that if $f$ and $g$ are the homeomorphisms of $X$ into $\overline{X}$ and $\bar{X}$ respectively, then $T \circ f = g$.) If we also have $\overline{X} \cong \bar{X}$, then $T$ is a homeomorphism, and in this case we write $\overline{X} \cong \bar{X}$.

2. $Q$-compactifications of $X$. Let $Q$ be a nonvoid family of continuous functions on $X$ with each $f \in Q$ having its range contained in a compact Hausdorff space $S_f$. Using the methods of [5], we now describe a compactification of $X$ which is the compactification defined in [5] when $X$ is locally compact.

Definition. Let $Y$ be the product space $\prod_{f \in Q} S_f$ and $e$ the evaluation map sending $X$ into $Y$. (For each $x \in X$, $e(x)(f) = f(x)$.) Set

$$\Delta = \cap \{e(X - K) : K \text{ compact, } K \subseteq X\},$$

and let $\overline{X}^Q$ be the (disjoint) union $X \cup \Delta$. Given an open set $U$ in $Y$ and a compact set $K \subseteq X$, we set $U_K = [U \cap \Delta] \cup [e^{-1}(U) - K]$. If $\mathfrak{T}$ is the topology on $\overline{X}^Q$ generated by the base consisting of all open
sets in $X$ and all the sets $U_k$, then $(\overline{X}^0, 3)$ is called the $Q$-compactification of $X$.\footnote{Added July 19, 1968. The author has learned of a manuscript, Minimum and maximum compactifications of arbitrary topological spaces, by R. F. Dickman, Jr., submitted in January 1967 to the Trans. Amer. Math. Soc. Using a different definition than the one given here and starting with an arbitrary topological space $X$ and a collection of mappings from $X$ into a single compact Hausdorff space $S$, Professor Dickman has proved the existence of a compactification $\alpha_0X$ which has the properties established for $\overline{X}^0$ by Theorems 2.1 and 2.2 below, and is thus equivalent to $\overline{X}^0$. Professor Dickman has informed the author that these results will be included in a revised paper entitled Compactifications and real-compactifications of arbitrary topological spaces.}

It is not hard to show that $\overline{X}^0$ is, indeed, a compactification of $X$; e.g., $\overline{X}^0$ is compact since a net which is eventually in the complement of every compact subset of $X$ has a cluster point in $\Delta$. Clearly, $\overline{X}^0$ also has the following properties:

**Theorem 2.1.** Each function $f \in Q$ has a continuous extension mapping $\overline{X}^0$ into $S_f$, and the family of these extensions separates the points in $\overline{X}^0 - X$. Moreover, $X$ is open in $\overline{X}^0$. Thus, if $X$ is not locally compact, $\overline{X}^0$ is neither Hausdorff nor a space which satisfies the first axiom of countability.

Next we show that these properties determine the compactification $\overline{X}^0$ up to a homeomorphism.

**Theorem 2.2.** Let $\overline{X}$ be a compactification of $X$ such that each function $f$ in a nonvoid subfamily $Q_0$ of $Q$ has a continuous extension mapping $\overline{X}$ into $S_f$ and these extensions separate the points in $\Delta \overline{X}$. Then $\overline{X}^0 \cong \overline{X}$. If, moreover, $X$ is open in $\overline{X}$ (e.g., if $X$ is locally compact) and if $Q_0 = Q$, then $\overline{X}^0 \cong \overline{X}$.

**Proof.** Let $\Gamma = \Delta \overline{X}$ and recall that $\Delta = \overline{X}^0 - X$. Let $e, \tilde{e}$, and $\tilde{e}$ be the evaluation maps sending $X$, $\overline{X}^0$, and $\overline{X}$ respectively into the product space $Y_0 = \prod_{f \in Q_0} S_f$. Given $x_0 \in \Delta$, let $N$ be a neighborhood of $\tilde{e}(x_0)$ in $Y_0$, and let $\overline{N}$ be its closure in $Y_0$. Then $\tilde{e}^{-1}(\overline{N}) \cap \Gamma \neq \emptyset$, for otherwise $e^{-1}(\overline{N}) = \tilde{e}^{-1}(\overline{N})$ is a compact subset of $X$, and $\tilde{e}^{-1}(N) - e^{-1}(\overline{N})$ is a neighborhood of $x_0$ in $\overline{X}^0$ that does not intersect $X$. Let $T(x_0)$ be the unique point in the intersection of all sets of the form $\tilde{e}^{-1}(\overline{N}) \cap \Gamma$ where $\overline{N}$ ranges over the neighborhood system of $\tilde{e}(x_0)$ in $Y_0$. One thus extends the identity mapping on $X$ to a function $T$ from $\overline{X}^0$ into $\overline{X}$, and in a similar way one shows that $T$ is onto. Clearly, $T$ is continuous at each point of $X$. Given $x_0 \in \Delta$ and $U$ an open neighborhood of $T(x_0)$ in $\overline{X}$, there is an open set $V$ in $Y_0$ such
that \( \bar{e}(\Gamma - U) \subseteq V \) and \( \bar{e}(T(x_0)) = e(x_0) \in \bar{V} \). Moreover, the set \( K = X - \{ U \cup e^{-1}(V) \} \) is a compact subset of \( X \). Let \( W = X^0 - [K \cup e^{-1}(V)] \). Then \( T(W) \subseteq U \), so \( T \) is continuous at \( x_0 \), and thus \( T \) is continuous on all of \( X^0 \). The rest of the proof is clear.

**Corollary 2.3.** If \( Q_0 \) is a nonvoid subset of \( Q \), then \( X^0 \supseteq X^{Q_0} \).

Let \( \bar{e} \) be the evaluation map sending \( X^Q \) into \( Y \); then

\[
\bar{e}(X^Q) = e(X),
\]

since \( \bar{e}(X^Q) = e(X) \cup \Delta \) in \( Y \). One can, moreover, easily establish the following result:

**Proposition 2.4.** If there are no compact neighborhoods in \( X \), then \( \Delta \) is the closure of \( e(X) \).

**Examples.** (1) If \( Q \) consists of one mapping to a one point space, then \( X^Q \) is the Alexandroff one point compactification of \( X \). (See [4, p. 150].)

(2) Let \( X \) be the rational numbers in the real unit interval \([0, 1]\), and let \( Q \) consist of the single function \( f(x) = x \). Then \( \Delta \) is homeomorphic to \([0, 1]\). A typical neighborhood of a point \( y_0 \in \Delta \) is given by a constant \( \epsilon > 0 \) and a compact subset \( K \) of \( X \); it has the form

\[
\{ y \in \Delta : |y - y_0| < \epsilon \} \cup \{ x \in X - K : |x - y_0| < \epsilon \}.
\]

If \( X = [0, 1] \), then \( X^0 \supseteq \bar{X} \), but we do not have \( X^0 \subseteq \bar{X} \).

(3) Let \( H \) be an infinite-dimensional Hilbert space with the norm topology and let \( H^* \) be its dual space with the weak topology. Let \( X \) be the closed unit ball in \( H \), \( Q \) be the functions in \( H^* \), and \( e \) be the canonical map sending \( H \) onto \( H^* \). By Proposition 2.4, \( \Delta = e(X) \), which is the closed unit ball with the weak topology. A typical neighborhood of a point \( x \in \Delta \) has the form \( [N \cap \Delta] \cup [e^{-1}(N) \cap X - K] \), where \( N \) is a weak neighborhood of \( x \) in \( H^* \) and \( K \subseteq X \) is compact in the norm topology.

Finally, we note that the results of this section can be applied to an arbitrary topological space \( X \) if one works with closed and compact subsets of \( X \) instead of compact subsets of \( X \). The details are left to the reader.

3. **Hausdorff \( Q \)-compactifications.** In this section, we assume that \( X \) is homeomorphic to its image \( e(X) \) in the product space \( Y = \prod_{f \in Q} S_f \) (see [4, p. 116]); \( Q \) is the family of functions described in §2. (If \( X \) is locally compact, then, following Constantinescu and Cornea [3], one may adjoin all continuous real-valued functions
with compact support to a given family of continuous functions to obtain a Q which satisfies these assumptions.) Identify X with e(X); as is well known [7], [2], [4] the closure of e(X) in Y is a compact Hausdorff space which contains X (i.e., e(X)) as a dense subset, and the functions in Q have continuous extensions to the closure of e(X). All the points in the closure of e(X) are separated by these extensions. We call this closure the Hausdorff Q-compactification of X, and we denote it by $\overline{X}^Q$. By Theorems 2.1 and 2.2, $\overline{X}^Q \supseteq \overline{X}^Q$, and $\overline{X}^Q \subseteq \overline{X}^Q$ if and only if X is locally compact. On the other hand, if there are no compact neighborhoods in X, then by Proposition 2.4, $\Delta = \Delta \overline{X}^Q$ is homeomorphic to $\overline{X}^Q$. The space $\overline{X}^Q$ is unique in the following sense:

**Theorem 3.1.** Let X be a Hausdorff compactification of X with each function in Q having a continuous extension mapping X into $\overline{X}$. If these extensions separate the points of $\overline{X} - X$, then $\overline{X}^Q = \overline{X}$.

**Proof.** We need only show that the evaluation map $\varepsilon$ which sends $\overline{X}$ onto $\overline{X}^Q$ is injective. Assume that $\varepsilon(x) = \varepsilon(y)$ for some $x \in X$ and $y \in \overline{X} - X$. Let $U \subseteq \overline{X}$ be a neighborhood of y such that $x \notin U$, and let $C = \overline{U} \cap X$. Then C is closed in X, so $\varepsilon(C) = X \cap D$ where D is closed in $\overline{X}^Q$. Since $\varepsilon(x)$ is not in D, y is not in the closed set $\varepsilon^{-1}(D)$. But this is impossible since y is in the closure of $C = \varepsilon^{-1}(D) \cap X$. Thus, $\varepsilon$ is injective and therefore a homeomorphism.

Note that if $Q_0$ is a nonvoid subset of Q and X is homeomorphic to its image in $\prod_{r \in Q_0} S_r$, then since the projection of $\prod_{r \in Q_0} S_r$ onto $\prod_{r \in Q_0} S_r$ is continuous, $\overline{X}^Q \supseteq \overline{X}^Q$.

**Examples.** (1) If X is the set of rational numbers in $[0, 1]$ and Q consists of the single function $f(x) = x$, then $\overline{X}^Q = [0, 1]$. (Compare with Example 2 of §2.)

(2) Let X be a metric space with metric d, and let Q be the family of functions $\{d_x: x \in X\}$, where $d_x(y) = d(x, y)$ for all $y \in X$. Each $d_x$ has its range in the interval $[0, +\infty]$. Set $X^* = \{z \in \overline{X}^Q: \exists \epsilon > 0 \exists x \in X \text{ with } d_x(z) < \epsilon\}$, and let $d^*(z, w) = \inf_{x \in X} [d_x(z) + d_x(w)]$ for each pair (z, w) in $X^* \times X^*$. Then one can show that $d^*$ is a metric which generates the relative product topology on $X^*$ and $(X^*, d^*)$ is the completion of $(X, d)$.

(3) A similar construction gives the completion $X^*$ of a Hausdorff uniform space X: If the uniform topology of X is generated by the family of pseudometrics $\{d_\alpha: \alpha \in A\}$, and $Q = \{d_\alpha(x, \cdot): \alpha \in A, x \in X\}$, then

$$X^* = \{z \in \overline{X}^Q: \forall \epsilon > 0 \text{ and } \forall \alpha \in A, \exists x \in X \text{ with } d_\alpha(x, z) < \epsilon\}.$$
Using filters, Samuel [6] has constructed the largest compactification $X^*$ in which a given uniform space $X$ can be uniformly imbedded; the completion $X^*$ is the subset of $X^*$ consisting of all limits of Cauchy ultrafilters in $X$. However if $Q$ is any collection of uniformly continuous functions from $X$ into the real unit interval $I$ such that $X^{HQ}$ exists, then $X$ is uniformly imbedded in $X^{HQ}$. Moreover, any Hausdorff compactification $X$ in which $X$ is uniformly imbedded is of the form $X^{HQ}$ where each $f \in Q$ maps $X$ uniformly into $I$. (See Theorem 4.2.) It follows that $X^* \cong X^{uQ}$ where $u$ is the set of all uniformly continuous mappings of $X$ into $I$. Thus the compactifications used in the last two examples are, in general, smaller than $X^*$. If, for example, $X$ is the real line with 0 removed and $X$ has the additive uniform structure, then the compactification used in Example 2 is the one point compactification of the real line where as $X^*$ is "a space almost as complicated as the Čech compactification of the real line" [6, p. 124].

4. Properties of arbitrary compactifications. Let $\hat{X}$ be any compactification of the Hausdorff space $X$ such that $\hat{X}$ satisfies the three conditions in §1. Let $R \subseteq \hat{X} \times \hat{X}$ be the equivalence relation which consists of the diagonal set $\{ (x, x) : x \in \hat{X} \}$ together with all pairs $(x, y) \in \hat{X} \times \hat{X}$ for which there is a $z \in \hat{X} - X$ such that neither $x$ nor $y$ can be separated from $z$ by disjoint open sets. As usual, $R[x]$ denotes the set of all points in $\hat{X}$ equivalent to a point $x$, and for any set $A \subseteq \hat{X}$, $R[A] = \bigcup_{x \in A} R[x]$.

**Proposition 4.1.** The relation $R$ has the following properties:

(i) For each $x \in \hat{X}$, $R[x]$ is closed and therefore compact.

(ii) If $x$ and $y$ are points in $\hat{X}$ with $R[x] \cap R[y] = \emptyset$, then there are disjoint open sets $U$ and $V$ in $\hat{X}$ with $R[x] \subseteq U$ and $R[y] \subseteq V$.

(iii) If $z \in \hat{X} - X$ and $U$ is an open neighborhood of $z$, then $R[z]$ is contained in the closure $\overline{U}$ of $U$.

(iv) If $x \in X \cap \Delta \hat{X}$, then $R[x] = \{ x \}$.

(v) If $C$ is compact in $\hat{X}$, then $R[C]$ is closed.

**Proof.** We shall only prove (v). We show first that $R[C] \cap \Delta \hat{X}$ is closed. If $\{z_\alpha\}_{\alpha \in A}$ is a net in $R[C] \cap \Delta \hat{X}$ and $\{z_\alpha\}$ converges to $z \in \Delta \hat{X}$, then for each $\alpha$ in the index set $A$ there is a point $x_\alpha$ in $R[z_\alpha] \cap C$. Let $x \in C$ be a cluster point of the net $\{x_\alpha\}_{\alpha \in A}$. Given open neighborhoods $U$ and $V$ of $z$ and $x$ respectively, there is an $\alpha \in A$ such that $z_\alpha \in U$ and $x_\alpha \in V$. Since $x_\alpha \in R[z_\alpha]$ and either $x_\alpha = z_\alpha$ or $z_\alpha \in \hat{X} - X$, it follows that $U \cap V \neq \emptyset$, and thus $z \in R[C]$. We have shown that $R[C] \cap \Delta \hat{X}$ is closed.
Assume now that $y_0 \in R[C]$. Then for each set $R[x] \subseteq R[C]$, there is a pair of disjoint open sets $U$ and $V$ in $X$ with $R[x] \subseteq U$ and $y_0 \in V$. Thus the compact set $\bigcup [R[C] \cap \Delta X]$ is contained in a finite union of open sets $\{U_i : i = 1, 2, \ldots, n\}$ such that $y_0 \notin \bigcup_{i=1}^n U_i$. But by (iii), $R[C]$ is contained in $\bigcup_{i=1}^n U_i$. Thus $R[C]$ is closed.

We next show that $X/R$ is Hausdorff; clearly, $R$ is the finest relation for which this can be true. It follows that the arbitrarily chosen compactification $\bar{X}$ is comparable with an appropriate $Q$-compactification. The following theorem for the case that $X$ is Hausdorff is due to Čech [2].

**Theorem 4.2.** Let $Q$ be the set of all mappings of $X$ into the unit interval $[0, 1]$, and let $R$ be the set of restrictions $\{f : X : f \in Q\}$. Then $Q$ separates the points in $\Delta X$, and thus $\bar{X}^Q \supseteq \bar{X}$. If $X$ is open in $\bar{X}$, then $\bar{X}^Q \cong \bar{X}$. If $X$ is Hausdorff, then $\bar{X}^Q \cong \bar{X}$.

**Proof.** By Theorems 2.2 and 3.1, we need only show that $Q$ separates the points in $\Delta X$. Let $P$ be the projection of $\bar{X}$ onto the quotient space $X/R$. If $P(x)$ and $P(y)$ are distinct points in $X/R$, then there are disjoint neighborhoods $U$ and $V$ of $R[x]$ and $R[y]$ in $X$. Let $C = X - U$ and $D = X - V$. Then $R[C]$ is a closed set with $R[C] \cap R[x] = \emptyset$; $R[D]$ is a closed set with $R[D] \cap R[y] = \emptyset$, and $R[D] \cup R[C] = \bar{X}$. Therefore, $P(R[C])$ and $P(R[D])$ are closed sets in $X/R$ with $P(x) \notin P(R[C])$, $P(y) \notin P(R[D])$, and $P(R[C]) \cup P(R[D]) = X/R$. Thus $X/R$ is Hausdorff, and the theorem follows from Urysohn’s lemma.

**Corollary 4.3.** Every compactification of a locally compact Hausdorff space is a $Q$-compactification. Every Hausdorff compactification of a completely regular space is a Hausdorff $Q$-compactification.

Finally, we let $s$ be the set of all mappings of $X$ into the unit interval $[0, 1]$, and we consider the compactifications $\bar{X}^s$ and $\bar{X}^{hs}$. Of course, $\bar{X}^{hs}$ is only defined if $X$ is completely regular, and it is the Stone-Čech compactification of $X$.

**Theorem 4.4.** Let $\bar{X}$ be any compactification of $X$. Then $\bar{X}^s \cong \bar{X}$, and as is well known, $\bar{X}^{hs} \cong \bar{X}$ if $\bar{X}$ is Hausdorff.

**Proof.** The result follows from Theorem 4.2, Corollary 2.3 and the remark following Theorem 3.1.

We have shown that if $X$ is completely regular, then $\bar{X}^s \cong \bar{X}^{hs}$, while $\bar{X} \cong \bar{X}^{hs}$ only if $X$ is locally compact. If $X$ is not locally compact, then $\bar{X}^s$ dominates a larger class of compactifications than the Stone-Čech compactification $\bar{X}^{hs}$. Indeed if $X$ is not completely
regular, $\overline{X^{he}}$ is not even defined. Moreover, $X$ is always an open subset of $\overline{X^e}$, but only when $X$ is locally compact is it open in $\overline{X^{he}}$.

References


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