COMPACTIFICATIONS OF HAUSDORFF SPACES

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1. Introduction. In this paper, we describe methods of imbedding a Hausdorff space $X$ in a compact space $\overline{X}$ so that each function in a given family of continuous functions on $X$ has a continuous extension to $\overline{X}$ and the family of extensions separates the points of $\overline{X} - X$. In particular, if $X$ is completely regular but not locally compact, then we shall exhibit a non-Hausdorff compactification which contains $X$ as an open subset and is bigger than the Stone-Cech compactification of $X$. (Of course, every compactification of $X$ is non-Hausdorff if $X$ is not completely regular.) We shall also show that the completion of a metric space $M$ may be obtained as a subset of a compactification of $M$ by a rather simple construction.

By a compactification of a Hausdorff space $X$, we mean a compact space $\overline{X}$ which contains, as a dense subset, the image of $X$ under a fixed homeomorphism $f$. We usually do not distinguish between $X$ and $f(X)$, and we say that $\overline{X}$ contains $X$ as a dense subset. In what follows, $X$ is always a noncompact Hausdorff space, $\Delta \overline{X}$ denotes the closure of $\overline{X} - X$ in $\overline{X}$, and a mapping is always a continuous function. If $\overline{X}$ is Hausdorff, we say that $\overline{X}$ is a Hausdorff compactification of $X$. If $\overline{X}$ is not Hausdorff, however, we still assume that it satisfies the following properties:

I. Compact subsets of $X$ are closed in $\overline{X}$.

II. Any two distinct points $x$ and $y$ in $\Delta \overline{X}$ can be separated by disjoint open sets; i.e., there exist open sets $U$ and $V$ in $\overline{X}$ with $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

III. For each point $x \in X$ there is at most one point $z \in \overline{X} - X$ such that $x$ and $z$ cannot be separated by disjoint open sets in $\overline{X}$.

Clearly, II and III are necessary conditions for the points in $\Delta \overline{X}$ to be separated by a family of continuous functions from $\overline{X}$ into a Hausdorff space; we shall show later that, together with Condition I, they are also sufficient. The following properties of $\overline{X}$ are consequences of I, II, and the fact that $X$ is dense in $\overline{X}$.

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Proposition 1.1. If $\overline{X}$ is a compactification of $X$, then:

(i) Two distinct points in $X$ can be separated by disjoint open sets in $\overline{X}$.
(ii) The space $\overline{X}$ is $T_1$.
(iii) In $\overline{X}$, a sequence converges to at most one point.
(iv) If $X$ satisfies the first axiom of countability then compact subsets of $\overline{X}$ are closed.
(v) If the compactification $\overline{X}$ satisfies the first axiom of countability, then $\overline{X}$ is Hausdorff.
(vi) $X$ is locally compact if and only if $X$ is open in $\overline{X}$ and $\overline{X}$ is Hausdorff.

Example. Following Arens [1], let $X$ be the set of all pairs of nonnegative integers such that each point other than $(0, 0)$ is an open set and every neighborhood of $(0, 0)$ contains all but a finite number of points in all but a finite number of columns $C_n$, where $C_n = \{(m, m) : m \in \mathbb{Z}^+\}$. Then $X$ is not first countable. Let $\overline{X}$ be the space $X$ together with a point $P$ whose neighborhoods omit at most a finite number of points in $X$. Any nonrepeating sequence with an infinite number of points in each column $C_n$ converges to $P$ and has cluster point $(0, 0)$. Moreover, $\overline{X}-\{(0, 0)\}$ is compact but not closed in $\overline{X}$.

Now let $\overline{X}$ and $\hat{X}$ be two compactifications of $X$. By the notation $\overline{X} \cong \hat{X}$ we mean there is a mapping $T$ of $\overline{X}$ onto $\hat{X}$ such that $T|X$ is the identity map. (To be more accurate, we should say that if $f$ and $g$ are the homeomorphisms of $X$ into $\overline{X}$ and $\hat{X}$ respectively, then $T \circ f = g$.) If we also have $\overline{X} \cong \hat{X}$, then $T$ is a homeomorphism, and in this case we write $\overline{X} \cong \hat{X}$.

2. $Q$-compactifications of $X$. Let $Q$ be a nonvoid family of continuous functions on $X$ with each $f \in Q$ having its range contained in a compact Hausdorff space $S_f$. Using the methods of [5], we now describe a compactification of $X$ which is the compactification defined in [5] when $X$ is locally compact.

Definition. Let $Y$ be the product space $\prod_{f \in Q} S_f$ and $e$ the evaluation map sending $X$ into $Y$. (For each $x \in X$, $e(x)(f) = f(x)$.) Set

$$\Delta = \cap \{e(X-K) : K \text{ compact}, K \subset X\},$$

and let $\overline{X}^Q$ be the (disjoint) union $X \cup \Delta$. Given an open set $U$ in $Y$ and a compact set $K \subset X$, we set $U_K = \overline{U \cap \Delta} \cup [e^{-1}(U) - K]$. If $\mathfrak{T}$ is the topology on $\overline{X}^Q$ generated by the base consisting of all open
sets in $X$ and all the sets $U_K$, then $(X^0, 3)$ is called the $Q$-compactification of $X$.

It is not hard to show that $X^0$ is, indeed, a compactification of $X$; e.g., $X^0$ is compact since a net which is eventually in the complement of every compact subset of $X$ has a cluster point in $\Delta$. Clearly, $X^0$ also has the following properties:

**Theorem 2.1.** Each function $f \in Q$ has a continuous extension mapping $X^0$ into $S_f$, and the family of these extensions separates the points in $X^0 - X$. Moreover, $X$ is open in $X^0$. Thus, if $X$ is not locally compact, $X^0$ is neither Hausdorff nor a space which satisfies the first axiom of countability.

Next we show that these properties determine the compactification $X^0$ up to a homeomorphism.

**Theorem 2.2.** Let $\bar{X}$ be a compactification of $X$ such that each function $f$ in a nonvoid subfamily $Q_0$ of $Q$ has a continuous extension mapping $X$ into $S_f$ and these extensions separate the points in $\Delta \bar{X}$. Then $X^0 \cong \bar{X}$. If, moreover, $X$ is open in $\bar{X}$ (e.g., if $X$ is locally compact) and if $Q_0 = Q$, then $X^0 \cong X$.

**Proof.** Let $\Gamma = \Delta \bar{X}$ and recall that $\Delta = X^0 - X$. Let $e$, $\bar{e}$, and $\bar{e}$ be the evaluation maps sending $X$, $X^0$, and $\bar{X}$ respectively into the product space $Y_0 = \prod_{f \in Q_0} S_f$. Given $x_0 \in \Delta$, let $N$ be a neighborhood of $\bar{e}(x_0)$ in $Y_0$, and let $\bar{N}$ be its closure in $Y_0$. Then $\bar{e}^{-1}(\bar{N}) \cap \Gamma \neq \emptyset$, for otherwise $e^{-1}(N) = \bar{e}^{-1}(\bar{N})$ is a compact subset of $X$, and $\bar{e}^{-1}(N) - e^{-1}(N)$ is a neighborhood of $x_0$ in $X^0$ that does not intersect $X$. Let $T(x_0)$ be the unique point in the intersection of all sets of the form $\bar{e}^{-1}(\bar{N}) \cap \Gamma$ where $N$ ranges over the neighborhood system of $\bar{e}(x_0)$ in $Y_0$. One thus extends the identity mapping on $X$ to a function $T$ from $X^0$ into $\bar{X}$, and in a similar way one shows that $T$ is onto. Clearly, $T$ is continuous at each point of $X$. Given $x_0 \in \Delta$ and $U$ an open neighborhood of $T(x_0)$ in $\bar{X}$, there is an open set $V$ in $Y_0$ such

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(Added July 19, 1968.) The author has learned of a manuscript, Minimum and maximum compactifications of arbitrary topological spaces, by R. F. Dickman, Jr., submitted in January 1967 to the Trans. Amer. Math. Soc. Using a different definition than the one given here and starting with an arbitrary topological space $X$ and a collection of mappings from $X$ into a single compact Hausdorff space $S$, Professor Dickman has proved the existence of a compactification $\alpha X$ which has the properties established for $X^0$ by Theorems 2.1 and 2.2 below, and is thus equivalent to $X^0$. Professor Dickman has informed the author that these results will be included in a revised paper entitled Compactifications and real-compactifications of arbitrary topological spaces.
that \( \bar{e}(\Gamma - U) \subseteq V \) and \( e(T(x_0)) = \bar{e}(x_0) \subseteq \bar{V} \). Moreover, the set 
\( K = X - [\cup \bar{e}^{-1}(V)] \) is a compact subset of \( X \). Let \( W = \overline{X^Q} - [K \cup \bar{e}^{-1}(V)] \). Then \( T(W) \subseteq U \), so \( T \) is continuous at \( x_0 \), and thus \( T \) is continuous on all of \( \overline{X^Q} \). The rest of the proof is clear. 

**Corollary 2.3.** If \( Q_0 \) is a nonvoid subset of \( Q \), then \( \overline{X^Q} \supseteq \overline{X^{Q_0}} \).

Let \( \bar{e} \) be the evaluation map sending \( \overline{X^Q} \) into \( Y \); then

\[ \bar{e}(\overline{X^Q}) = \bar{e}(X) \]

since \( \bar{e}(\overline{X^Q}) = e(X) \cup \Delta \) in \( Y \). One can, moreover, easily establish the following result:

**Proposition 2.4.** If there are no compact neighborhoods in \( X \), then \( \Delta \) is the closure of \( e(X) \).

**Examples.** (1) If \( Q \) consists of one mapping to a one point space, then \( \overline{X^Q} \) is the Alexandroff one point compactification of \( X \). (See [4, p. 150].)

(2) Let \( X \) be the rational numbers in the real unit interval \([0, 1]\), and let \( Q \) consist of the single function \( f(x) = x \). Then \( \Delta \) is homeomorphic to \([0, 1]\). A typical neighborhood of a point \( y_0 \in \Delta \) is given by a constant \( \epsilon > 0 \) and a compact subset \( K \) of \( X \); it has the form

\[ \{ y \in \Delta : |y - y_0| < \epsilon \} \cup \{ x \in X - K : |x - y_0| < \epsilon \} \]

If \( X = [0, 1] \), then \( \overline{X^Q} \supseteq \bar{X} \), but we do not have \( \overline{X^Q} \subseteq \bar{X} \).

(3) Let \( H \) be an infinite-dimensional Hilbert space with the norm topology and let \( H^* \) be its dual space with the weak topology. Let \( X \) be the closed unit ball in \( H \), \( Q \) be the functions in \( H^* \), and \( e \) be the canonical map sending \( H \) onto \( H^* \). By Proposition 2.4, \( \Delta = e(X) \), which is the closed unit ball with the weak topology. A typical neighborhood of a point \( x \in \Delta \) has the form \( [N \cap \Delta] \cup [e^{-1}(N) \cap X - K] \), where \( N \) is a weak neighborhood of \( x \) in \( H^* \) and \( K \subseteq X \) is compact in the norm topology.

Finally, we note that the results of this section can be applied to an arbitrary topological space \( X \) if one works with closed and compact subsets of \( X \) instead of compact subsets of \( X \). The details are left to the reader.

3. **Hausdorff \( Q \)-compactifications.** In this section, we assume that \( X \) is homeomorphic to its image \( e(X) \) in the product space \( Y = \prod_{f \in Q} S_f \) (see [4, p. 116]); \( Q \) is the family of functions described in §2. (If \( X \) is locally compact, then, following Constantinescu and Cornea [3], one may adjoin all continuous real-valued functions
with compact support to a given family of continuous functions to obtain a $Q$ which satisfies these assumptions.) Identify $X$ with $e(X)$; as is well known [7], [2], [4] the closure of $e(X)$ in $Y$ is a compact Hausdorff space which contains $X$ (i.e., $e(X)$) as a dense subset, and the functions in $Q$ have continuous extensions to the closure of $e(X)$. All the points in the closure of $e(X)$ are separated by these extensions. We call this closure the Hausdorff $Q$-compactification of $X$, and we denote it by $X^Q$. By Theorems 2.1 and 2.2, $X^Q \supseteq X^Q$, and $X^Q = X^Q$ if and only if $X$ is locally compact. On the other hand, if there are no compact neighborhoods in $X$, then by Proposition 2.4, $\Delta = \Delta X^Q$ is homeomorphic to $X^Q$. The space $X^Q$ is unique in the following sense:

**THEOREM 3.1.** Let $X$ be a Hausdorff compactification of $X$ with each function in $Q$ having a continuous extension mapping $X$ into $S_f$. If these extensions separate the points of $X - X$, then $X^Q = X$.

**Proof.** We need only show that the evaluation map $\bar{e}$ which sends $X$ onto $X^Q$ is injective. Assume that $\bar{e}(x) = \bar{e}(y)$ for some $x \in X$ and $y \in X - X$. Let $U \subseteq X$ be a neighborhood of $y$ such that $x \notin U$, and let $C = U \cap X$. Then $C$ is closed in $X$, so $\bar{e}(C) = X \cap X$ where $D$ is closed in $X^Q$. Since $\bar{e}(x)$ is not in $D$, $y$ is not in the closed set $\bar{e}^{-1}(D)$. But this is impossible since $y$ is in the closure of $C = \bar{e}^{-1}(D) \cap X$. Thus, $\bar{e}$ is injective and therefore a homeomorphism.

Note that if $Q_0$ is a nonvoid subset of $Q$ and $X$ is homeomorphic to its image in $\prod_{\alpha \in Q} S_f$, then since the projection of $\prod_{\alpha \in Q} S_f$ onto $\prod_{\alpha \in Q_0} S_f$ is continuous, $X^Q \supseteq X^Q$.

**EXAMPLES.** (1) If $X$ is the set of rational numbers in $[0, 1]$ and $Q$ consists of the single function $f(x) = x$, then $X^Q = [0, 1]$. (Compare with Example 2 of §2.)

(2) Let $X$ be a metric space with metric $d$, and let $Q$ be the family of functions $\{d_x : x \in X\}$, where $d_y(x) = d(x, y)$ for all $y \in X$. Each $d_x$ has its range in the interval $[0, +\infty]$. Set $X^* = \{z \in X^Q : \forall \epsilon > 0 \exists x \in X$ with $d_z(x) < \epsilon\}$, and let $d^*(z, w) = \inf_{x \in X}[d_z(x) + d_z(w)]$ for each pair $(z, w)$ in $X^* \times X^*$. Then one can show that $d^*$ is a metric which generates the relative product topology on $X^*$ and $(X^*, d^*)$ is the completion of $(X, d)$.

(3) A similar construction gives the completion $X^*$ of a Hausdorff uniform space $X$ if the uniform topology of $X$ is generated by the family of pseudometrics $\{d_\alpha : \alpha \in A\}$, and $Q = \{d_\alpha(x, \cdot) : \alpha \in A, x \in X\}$, then

$$X^* = \{z \in X^Q : \forall \epsilon > 0 \text{ and } \forall \alpha \in A, \exists x \in X \text{ with } d_\alpha(x, z) < \epsilon\}.$$
Using filters, Samuel [6] has constructed the largest compactification $X^*$ in which a given uniform space $X$ can be uniformly imbedded; the completion $X^*$ is the subset of $X^*$ consisting of all limits of Cauchy ultrafilters in $X$. However if $Q$ is any collection of uniformly continuous functions from $X$ into the real unit interval $I$ such that $X^{\text{Hq}}$ exists, then $X$ is uniformly imbedded in $X^{\text{Hq}}$. Moreover, any Hausdorff compactification $\tilde{X}$ in which $X$ is uniformly imbedded is of the form $\tilde{X}^{\text{Hq}}$ where each $f \in Q$ maps $X$ uniformly into $I$. (See Theorem 4.2.) It follows that $X^* \cong \tilde{X}^{\text{Hq}}$, where $\mathfrak{u}$ is the set of all uniformly continuous mappings of $X$ into $I$. Thus the compactifications used in the last two examples are, in general, smaller than $X^*$. If, for example, $X$ is the real line with $0$ removed and $X$ has the additive uniform structure, then the compactification used in Example 2 is the one point compactification of the real line where as $X^*$ is “a space almost as complicated as the Čech compactification of the real line” [6, p. 124].

4. Properties of arbitrary compactifications. Let $\tilde{X}$ be any compactification of the Hausdorff space $X$ such that $\tilde{X}$ satisfies the three conditions in §1. Let $R \subseteq \tilde{X} \times \tilde{X}$ be the equivalence relation which consists of the diagonal set $\{ (x, x) : x \in \tilde{X} \}$ together with all pairs $(x, y) \in \tilde{X} \times \tilde{X}$ for which there is a $z \in \tilde{X} - X$ such that neither $x$ nor $y$ can be separated from $z$ by disjoint open sets. As usual, $R[x]$ denotes the set of all points in $\tilde{X}$ equivalent to a point $x$, and for any set $A \subseteq \tilde{X}$, $R[A] = \bigcup_{x \in A} R[x]$.

Proposition 4.1. The relation $R$ has the following properties:

(i) For each $x \in \tilde{X}$, $R[x]$ is closed and therefore compact.
(ii) If $x$ and $y$ are points in $\tilde{X}$ with $R[x] \cap R[y] = \emptyset$, then there are disjoint open sets $U$ and $V$ in $\tilde{X}$ with $R[x] \subseteq U$ and $R[y] \subseteq V$.
(iii) If $z \in \tilde{X} - X$ and $U$ is an open neighborhood of $z$, then $R[z]$ is contained in the closure $\overline{U}$ of $U$.
(iv) If $x \in X \cap \Delta \tilde{X}$, then $R[x] = \{ x \}$.
(v) If $C$ is compact in $\tilde{X}$, then $R[C]$ is closed.

Proof. We shall only prove (v). We show first that $R[C] \cap \Delta \tilde{X}$ is closed. If $\{ z_\alpha \}_{\alpha \in A}$ is a net in $R[C] \cap \Delta \tilde{X}$ and $\{ z_\alpha \}$ converges to $z \in \Delta \tilde{X}$, then for each $\alpha$ in the index set $A$ there is a point $x_\alpha$ in $R[z_\alpha] \cap C$. Let $x \in C$ be a cluster point of the net $\{ x_\alpha \}_{\alpha \in A}$. Given open neighborhoods $U$ and $V$ of $z$ and $x$ respectively, there is an $\alpha \in A$ such that $z_\alpha \in U$ and $x_\alpha \in V$. Since $x_\alpha \in R[z_\alpha]$ and either $x_\alpha = z_\alpha$ or $z_\alpha \in \tilde{X} - X$, it follows that $U \cap V \neq \emptyset$, and thus $z \in R[C]$. We have shown that $R[C] \cap \Delta \tilde{X}$ is closed.
Assume now that $y_0 \in R[C]$. Then for each set $R[x] \subseteq R[C]$, there is a pair of disjoint open sets $U$ and $V$ in $X$ with $R[x] \subseteq U$ and $y_0 \in V$.

Thus the compact set $C \cup [R[C] \cap \Delta X]$ is contained in a finite union of open sets $\{ U_i : i = 1, 2, \ldots, n \}$ such that $y_0 \in \bigcup_{i=1}^{n} U_i$. But by (iii), $R[C]$ is contained in $\bigcup_{i=1}^{n} U_i$. Thus $R[C]$ is closed.

We next show that $X/R$ is Hausdorff; clearly, $R$ is the finest relation for which this can be true. It follows that the arbitrarily chosen compactification $\check{X}$ is comparable with an appropriate $Q$-compactification. The following theorem for the case that $X$ is Hausdorff is due to Čech [2].

**Theorem 4.2.** Let $\check{Q}$ be the set of all mappings of $X$ into the unit interval $[0, 1]$, and let $Q$ be the set of restrictions $\{ f : X \rightarrow \check{Q} \}$. Then $\check{Q}$ separates the points in $\Delta X$, and thus $\check{X} \cong \check{X}$. If $X$ is open in $\check{X}$, then $\check{X} \cong \check{X}$. If $X$ is Hausdorff, then $\check{X} \cong \check{X}$.

**Proof.** By Theorems 2.2 and 3.1, we need only show that $\check{Q}$ separates the points in $\Delta X$. Let $P$ be the projection of $\check{X}$ onto the quotient space $X/R$. If $P(x)$ and $P(y)$ are distinct points in $X/R$, then there are disjoint neighborhoods $U$ and $V$ of $R[x]$ and $R[y]$ in $X$. Let $C = X - U$ and $D = X - V$. Then $R[C]$ is a closed set with $R[C] \cap R[x] = \emptyset$; $R[D]$ is a closed set with $R[D] \cap R[y] = \emptyset$, and $R[D] \cup R[C] = \check{X}$. Therefore, $P(R[C])$ and $P(R[D])$ are closed sets in $\check{X}/R$ with $P(x) \notin P(R[C])$, $P(y) \notin P(R[D])$, and $P(R[C]) \cup P(R[D]) = \check{X}/R$. Thus $\check{X}/R$ is Hausdorff, and the theorem follows from Urysohn’s lemma.

**Corollary 4.3.** Every compactification of a locally compact Hausdorff space is a $Q$-compactification. Every Hausdorff compactification of a completely regular space is a Hausdorff $Q$-compactification.

Finally, we let $\mathcal{S}$ be the set of all mappings of $X$ into the unit interval $[0, 1]$, and we consider the compactifications $\check{X}^s$ and $\check{X}^{hs}$. Of course, $\check{X}^{hs}$ is only defined if $X$ is completely regular, and it is the Stone-Čech compactification of $X$.

**Theorem 4.4.** Let $\check{X}$ be any compactification of $X$. Then $\check{X}^s \cong \check{X}$, and as is well known, $\check{X}^{hs} \cong \check{X}$ if $\check{X}$ is Hausdorff.

**Proof.** The result follows from Theorem 4.2, Corollary 2.3 and the remark following Theorem 3.1.
regular, $\overline{X}^{\theta*}$ is not even defined. Moreover, $X$ is always an open subset of $\overline{X}^*$, but only when $X$ is locally compact is it open in $\overline{X}^{\theta*}$.

**References**


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