

THE STRUCTURE OF O'/Ω OVER LOCAL FIELDS OF CHARACTERISTIC 2

EDWARD A. CONNORS

In this paper V will be a nondefective quadratic space over a field of characteristic 2, $O(V)$ will be the orthogonal group of V , $O^+(V)$ will be the group of rotations, $\Omega(V)$ will be the commutator subgroup of $O(V)$ and $O'(V)$ will be the spinorial kernel (considered as a subgroup of O^+). We will show that $O' = \Omega$ if F is a local field. This result is known if V is isotropic and may be found in [3].

We assume as familiar the theory of quadratic forms over fields of characteristic 2 as treated in [1] and [3]. In §1 we state some basic results about the general theory and the local theory.

The word "field" will mean "field of characteristic 2." When possible we use the notation and terminology of [5]. The matrix notation of [7] will also be employed.

1. Preliminaries. Let x be an anisotropic vector in V . The orthogonal transvection with respect to the anisotropic line Fx will be denoted by τ_x . Dieudonné has shown in [4] that each σ in $O(V)$ may be expressed as $\sigma = \tau_{x_1} \cdots \tau_{x_r}$, where $r \leq \dim V = n$ and V is not a quaternary hyperbolic space over F_2 . In §1 we assume that $F \neq F_2$ in order to avoid this exceptional case. In §2 the case $F = F_2$ is precluded by the assumption that F is a local field.

The method of proof used to establish 43.6 in [5] may be employed to show that Ω is generated by elements of the type $\tau_x \tau_y \tau_x \tau_y$. Thus $\Omega \subseteq O^+$ and $\Omega \subseteq O'$.

The group O^+ may be characterized as the set of all elements of O which have an expression as a product of exactly n orthogonal transvections. Replacing the role of the discriminant by the Dickson invariant in the proof of 43.3b in [5] will establish that each σ in O^+ has an expression as a product of n orthogonal transvections where the first (or last) is chosen arbitrarily. In particular, any σ in O' or Ω has such an expression.

PROPOSITION 1. *Let V be a nondefective quadratic space of dimension n over the field F .*

(i) *If $n = 2$ then $O'(V) = \Omega(V)$.*

(ii) *If U is a nondefective subspace of V , the groups $O'(U)$ and $\Omega(U)$*

Received by the editors November 29, 1968.

may be considered in a natural way as subgroups of $O'(V)$ and $\Omega(V)$ respectively.

(iii) If $\{u_1, \dots, u_r\}$ and $\{v_1, \dots, v_r\}$ are sets of anisotropic vectors in V and $\{Q(u_1), \dots, Q(u_r)\}$ is a permutation of $\{Q(v_1), \dots, Q(v_r)\}$ then $\tau_{u_1} \dots \tau_{u_r} \in \tau_{v_1} \dots \tau_{v_r} \Omega$.

(iv) Let $\sigma \in O'(V)$ and assume that $\sigma = \prod_{i=1}^{2r} \tau_{x_i}$. If V contains a nondefective subspace U with $\Omega(U) = O'(U)$ and $Q(x_i) \in Q(U)$, $1 \leq i \leq 2r$, then $\sigma \in \Omega(V)$.

PROOF. The proofs of these statements are the same as those used in [5] to prove the analogous results in the characteristic not 2 theory.

Now let F be a local field, \mathfrak{o} the integers of F , \mathfrak{u} the units of \mathfrak{o} , \mathfrak{p} the maximal ideal of \mathfrak{o} , π a fixed prime element and \bar{F} the residue class field of F . Since \bar{F} is a finite field, the set $\{\alpha^2 + \alpha \mid \alpha \in \bar{F}\} = \wp(\bar{F})$ is a subgroup of index 2 in the additive group of \bar{F} . We let 0 and ρ be representatives of $\bar{F}/\wp(\bar{F})$.

PROPOSITION 2. Let V be a nondefective quadratic space over a local field F .

(i) If

$$V \cong \begin{pmatrix} & 1 \\ a & a^{-1}\rho \end{pmatrix}$$

then $Q(V) = a\mathfrak{u}F^2$.

(ii) If V is quaternary anisotropic then

$$V \cong \begin{pmatrix} & 1 \\ 1 & \rho \end{pmatrix} \perp \begin{pmatrix} & 1 \\ \pi & \pi^{-1}\rho \end{pmatrix}$$

(iii) V is universal if $\dim V \geq 4$ and isotropic if $\dim V \geq 6$.

(iv) If V is quaternary and P is binary with $\Delta P \neq \Delta V$ then $P \rightarrow V$. In particular if P and V are anisotropic then $P \rightarrow V$.

PROOF. See [6].

2. The main result.

LEMMA 1. Let F be a local field. Let ϵ be a fixed nonsquare unit. Then there exists a nondefective binary, anisotropic space over F which represents 1, π , $\pi\epsilon$ and ϵ .

PROOF.¹ The quadratic abelian extensions of F are characterized by the property that they are splitting fields of irreducible polynomials of the type $X^2 + X + d$, $d \in F$. Let E be such an extension.

¹ The referee has provided an excellent modification of my original proof. I wish to acknowledge his suggestion and to thank him for it.

Then $E = F(\delta)$ where $\delta^2 + \delta + d = 0$. Consider the nondefective binary space over F given by the matrix

$$P \cong \begin{pmatrix} & 1 \\ 1 & d \end{pmatrix}.$$

Clearly, P is anisotropic and $Q(P) = N_{E|F}(E)$. Thus we are done if we produce a quadratic abelian extension E over F whose norm group $N_{E|F}(\hat{E})$ contains the desired elements. By the existence theorem of local class field theory, see [2], it is enough to produce an open subgroup of index 2 in \hat{F} which contains the elements $1, \pi, \epsilon$ and $\pi\epsilon$. This we now do.

Let $\mathfrak{u}^{(n)} = \{\epsilon \in \mathfrak{u} \mid \epsilon \equiv 1 \pmod{\pi^n}\}$. Thus $\mathfrak{u}^{(0)} = \mathfrak{u}$ and $\mathfrak{u}^{(0)} \supset \mathfrak{u}^{(1)} \supset \dots$. Now $\mathfrak{u}/\mathfrak{u}^{(1)} \cong \bar{F}$ and $\mathfrak{u}^{(i)}/\mathfrak{u}^{(i+1)} \cong \bar{F}^+$ for $i \geq 1$. Thus $\mathfrak{u}/\mathfrak{u}^{(4)}$ has order $(q-1)q^3$ where q is the cardinality of \bar{F} . Splitting $\mathfrak{u}/\mathfrak{u}^{(4)}$ into its unique p -primary components, we may write $\mathfrak{u}/\mathfrak{u}^{(4)} = A \oplus B$ where the order of $A = q-1$ and the order of $B = q^3$. Since q is a power of 2, the order of B is a power of 2 and at least 8. Since $\mathfrak{u}^{(1)}/\mathfrak{u}^{(4)}$ is a subgroup of $\mathfrak{u}/\mathfrak{u}^{(4)}$ of order q^3 , $B = \mathfrak{u}^{(1)}/\mathfrak{u}^{(4)}$. Since $\lambda^4 \in \mathfrak{u}^{(4)}$ for any $\lambda \in \mathfrak{u}^{(1)}$, B is not cyclic. Thus any element of B is contained in a subgroup C of index 2 in B .

Let ϵ be the nonsquare unit under consideration. The image of ϵ under the isomorphism from $\mathfrak{u}/\mathfrak{u}^{(4)}$ to $A \oplus B$ is contained in a subgroup $A \oplus C$ where C is of the type described above. Clearly $A \oplus C$ has index 2 in $A \oplus B$. Thus ϵ is contained in a subgroup G of \mathfrak{u} of index 2 with $G \supset \mathfrak{u}^{(4)}$. But then G is open in \mathfrak{u} and, since \mathfrak{u} is open in \hat{F} , G is open in \hat{F} . Denoting by $\langle \pi \rangle$ the cyclic subgroup of \hat{F} generated by π , we see that $G\langle \pi \rangle = \bigcup_{n \in \mathbb{Z}} G\pi^n$ is also open in \hat{F} and $(\hat{F}:G\langle \pi \rangle) = (\mathfrak{u}\langle \pi \rangle : G\langle \pi \rangle) = 2$. Thus $G\langle \pi \rangle$ is an open subgroup of \hat{F} of index 2 that contains $1, \pi, \epsilon$ and $\pi\epsilon$. Q.E.D.

THEOREM. *If V is a nondefective quadratic space over the local field F then $O'(V) = \Omega(V)$.*

PROOF. In light of 1(i), 2(iii) and the results of [3] we may assume that V is quaternary anisotropic. Hence $V = P_1 \perp P_2$ where

$$P_1 \cong \begin{pmatrix} & 1 \\ 1 & \rho \end{pmatrix} \quad \text{and} \quad P_2 \cong \begin{pmatrix} & 1 \\ \pi & \pi^{-1}\rho \end{pmatrix}.$$

Let $\sigma \in O'(V)$. The universality of V and the results of §1 allow us to assume that $\sigma = \tau_{x_1}\tau_{x_2}\tau_{x_3}\tau_{x_4}$ where $Q(x_1) = 1$. Since $\tau_x = \tau_{\alpha x}$, $\alpha \in \hat{F}$, we may assume that $Q(x_i) = \pi$ or $\pi\epsilon_i$, $\epsilon_i \in \mathfrak{u}$, $1 \leq i \leq 4$. Since $\prod_{i=1}^4 Q(x_i) \in F^2$ we have two cases to consider:

(a) $Q(x_i) \in \mathfrak{u}$, $1 \leq i \leq 4$.

In this case $Q(x_i) \in Q(P_1)$ by Proposition 2(i). Now we apply 1(iv) and 1(i).

(b) Exactly two of the $Q(x_i)$, $1 \leq i \leq 4$, are of the form $Q(x_i) = \pi\epsilon_i$, $\epsilon_i \in \mathfrak{u}$. We may assume that $Q(x_2) = \epsilon_2$, $Q(x_3) = \pi\epsilon_3$ and $Q(x_4) = \pi\epsilon_4$ by Proposition 1(iii).

Since V is universal we may select y_1 and y_2 in V with $Q(y_1) = \pi$ and $Q(y_2) = \pi\epsilon_2$. Let $\Sigma = \tau_{y_1}\tau_{y_2}\tau_{x_3}\tau_{x_4}$. Clearly $\Sigma \in O'(V)$. Moreover, by applying 2(i), 1(i) and 1(iv) to P_2 we see that $\Sigma \in \Omega(V)$. If $\sigma\Sigma \in \Omega(V)$ we will be done. By applying 1(iii) we see that $\sigma\Sigma \in \Omega(V)$ if $\tau_{x_1}\tau_{x_2}\tau_{y_1}\tau_{y_2} \in \Omega(V)$. If $\epsilon_2 \in \mathfrak{u}^2$ then $\tau_{x_1}\tau_{x_2}$ and $\tau_{y_1}\tau_{y_2}$ are in $O'(V)$. By applying 1(iv) to the spaces P_1 and P_2 respectively, we see that $\tau_{x_1}\tau_{x_2}$ and $\tau_{y_1}\tau_{y_2}$ are in $\Omega(V)$. Hence $\tau_{x_1}\tau_{x_2}\tau_{y_1}\tau_{y_2} \in \Omega(V)$. Thus we may assume that $\epsilon_2 \notin \mathfrak{u}^2$. By the above lemma there is a nondefective, binary, anisotropic space P which represents 1 , π , $\pi\epsilon_2$ and ϵ_2 . We have $P \rightarrow V$ by 2(iv). Thus V contains a nondefective binary space B which represents 1 , π , $\pi\epsilon_2$ and ϵ_2 . Applying 1(iv) and 1(i) to B yields the desired result. Q.E.D.

REFERENCES

1. C. Arf, *Untersuchungen über quadratische Formen in Körpern der Charakteristik 2*, J. Reine Angew. Math. **183** (1941), 148–167.
2. J. W. S. Cassels and A. Fröhlich, editors, *Algebraic number theory*, Thompson Book Co., Washington, D.C., 1967.
3. J. Dieudonné, *La géométrie des groupes classiques*, Springer-Verlag, Berlin, 1963.
4. ———, *Sur les générateurs des groupes classiques*, Summa Brasil. Math. **3** (1955), 149–178.
5. O. T. O'Meara, *Introduction to quadratic forms*, Springer-Verlag, Berlin, 1963.
6. C. R. Riehm, *Integral representation of quadratic forms in characteristic 2*, Amer. J. Math. **86** (1964), 25–62.
7. C. H. Sah, *Quadratic forms over fields of characteristic 2*, Amer. J. Math. **82** (1960), 812–830.

UNIVERSITY OF NOTRE DAME