THE STRUCTURE OF $O'/\Omega$ OVER LOCAL FIELDS OF CHARACTERISTIC 2

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In this paper $V$ will be a nondefective quadratic space over a field of characteristic 2, $O(V)$ will be the orthogonal group of $V$, $O^+(V)$ will be the group of rotations, $\Omega(V)$ will be the commutator subgroup of $O(V)$ and $O'(V)$ will be the spinorial kernel (considered as a subgroup of $O^+$). We will show that $O' = \Omega$ if $F$ is a local field. This result is known if $V$ is isotropic and may be found in [3].

We assume as familiar the theory of quadratic forms over fields of characteristic 2 as treated in [1] and [3]. In §1 we state some basic results about the general theory and the local theory.

The word "field" will mean "field of characteristic 2." When possible we use the notation and terminology of [5]. The matrix notation of [7] will also be employed.

1. Preliminaries. Let $x$ be an anisotropic vector in $V$. The orthogonal transvection with respect to the anisotropic line $F_x$ will be denoted by $t_x$. Dieudonné has shown in [4] that each $\sigma$ in $O(V)$ may be expressed as $\sigma = \tau_{x_1} \cdots \tau_{x_r}$, where $r \leq \dim V = n$ and $V$ is not a quaternary hyperbolic space over $F_2$. In §1 we assume that $F \neq F_2$ in order to avoid this exceptional case. In §2 the case $F = F_2$ is precluded by the assumption that $F$ is a local field.

The method of proof used to establish 43.6 in [5] may be employed to show that $O$ is generated by elements of the type $t_x t_y t_x t_y$. Thus $\Omega \subseteq O^+$ and $\Omega \subseteq O'$.

The group $O^+$ may be characterized as the set of all elements of $O$ which have an expression as a product of exactly $n$ orthogonal transvections. Replacing the role of the discriminant by the Dickson invariant in the proof of 43.3b in [5] will establish that each $\sigma$ in $O^+$ has an expression as a product of $n$ orthogonal transvections where the first (or last) is chosen arbitrarily. In particular, any $\sigma$ in $O'$ or $\Omega$ has such an expression.

**Proposition 1.** Let $V$ be a nondefective quadratic space of dimension $n$ over the field $F$.

(i) If $n = 2$ then $O'(V) = \Omega(V)$.

(ii) If $U$ is a nondefective subspace of $V$, the groups $O'(U)$ and $\Omega(U)$...
may be considered in a natural way as subgroups of \( O'(V) \) and \( \Omega(V) \) respectively.

(iii) If \( \{u_1, \ldots, u_r\} \) and \( \{v_1, \ldots, v_r\} \) are sets of anisotropic vectors in \( V \) and \( \{Q(u_1), \ldots, Q(u_r)\} \) is a permutation of \( \{Q(v_1), \ldots, Q(v_r)\} \) then \( \tau_{u_1} \cdots \tau_{u_r} \in \tau_{v_1} \cdots \tau_{v_r} \Omega \).

(iv) Let \( \sigma \in O'(V) \) and assume that \( \sigma = \prod_{i=1}^{2r} \tau_{e_i} \). If \( V \) contains a nondefective subspace \( U \) with \( \Omega(U) = O'(U) \) and \( Q(x_i) \in \Omega(U), 1 \leq i \leq 2r \), then \( \sigma \in \Omega(V) \).

**Proof.** The proofs of these statements are the same as those used in [5] to prove the analogous results in the characteristic not 2 theory.

Now let \( F \) be a local field, \( \mathfrak{o} \) the integers of \( F \), \( \mathfrak{u} \) the units of \( \mathfrak{o} \), \( \mathfrak{p} \) the maximal ideal of \( \mathfrak{o} \), \( \pi \) a fixed prime element and \( \mathcal{F} \) the residue class field of \( F \). Since \( \mathcal{F} \) is a finite field, the set \( \{x^2 + x \mid x \in F\} = \mathcal{Q}(\mathcal{F}) \) is a subgroup of index 2 in the additive group of \( \mathcal{F} \). We let 0 and \( \rho \) be representatives of \( \mathcal{F} / \mathcal{Q}(\mathcal{F}) \).

**Proposition 2.** Let \( V \) be a nondefective quadratic space over a local field \( F \).

(i) If 
\[
V \cong \left( \begin{array}{c} 1 \\ a^{-1} \rho \end{array} \right)
\]
then \( Q(V) = auF^2 \).

(ii) If \( V \) is quaternary anisotropic then 
\[
V \cong \left( \begin{array}{c} 1 \\ \rho \end{array} \right) \perp \left( \begin{array}{c} 1 \\ \pi^{-1} \rho \end{array} \right)
\]

(iii) \( V \) is universal if \( \dim V \geq 4 \) and isotropic if \( \dim V \geq 6 \).

(iv) If \( V \) is quaternary and \( P \) is binary with \( \Delta P \neq \Delta V \) then \( P \to V \).
In particular if \( P \) and \( V \) are anisotropic then \( P \to V \).

**Proof.** See [6].

2. **The main result.**

**Lemma 1.** Let \( F \) be a local field. Let \( \epsilon \) be a fixed nonsquare unit. Then there exists a nondefective binary, anisotropic space over \( F \) which represents 1, \( \pi \), \( \pi \epsilon \) and \( \epsilon \).

**Proof.**

The quadratic abelian extensions of \( F \) are characterized by the property that they are splitting fields of irreducible polynomials of the type \( X^2 + X + d, d \in F \). Let \( E \) be such an extension.

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1 The referee has provided an excellent modification of my original proof. I wish to acknowledge his suggestion and to thank him for it.
Then $E = F(\delta)$ where $\delta^2 + \delta + d = 0$. Consider the nondefective binary space over $\hat{F}$ given by the matrix

$$P \cong \begin{pmatrix} 1 \\ 1 \\ d \end{pmatrix}.$$ 

Clearly, $P$ is anisotropic and $Q(P) = N_{\hat{F}/F}(E)$. Thus we are done if we produce a quadratic abelian extension $E$ over $F$ whose norm group $N_{\hat{F}/F}(E)$ contains the desired elements. By the existence theorem of local class field theory, see [2], it is enough to produce an open subgroup of index 2 in $\hat{F}$ which contains the elements $1, \pi, \epsilon$ and $\pi \epsilon$. This we now do.

Let $u(\alpha) = \{ e^uu | e \equiv 1 \mod \pi^n \}$. Thus $u^{(0)} = u$ and $u^{(i)} \supset u^{(i+1)} \supset \cdots$.

Now $u/u^{(1)} \cong \hat{F}^*$ and $u^{(i)}/u^{(i+1)} \cong \hat{F}^*$ for $i \geq 1$. Thus $u/u^{(4)}$ has order $(q-1)q^2$ where $q$ is the cardinality of $\hat{F}$. Splitting $u/u^{(4)}$ into its unique $p$-primary components, we may write $u/u^{(4)} = A \oplus B$ where the order of $A = q-1$ and the order of $B = q^2$. Since $q$ is a power of 2, the order of $B$ is a power of 2 and at least 8. Since $u^{(i)}/u^{(4)}$ is a subgroup of $u/u^{(4)}$ of order $q^2$, $B = u^{(1)}/u^{(4)}$. Since $u^{(4)}$ is a subgroup of $u/u^{(4)}$, $B$ is not cyclic. Thus any element of $B$ is contained in a subgroup $C$ of index 2 in $B$.

Let $\epsilon$ be the nonsquare unit under consideration. The image of $\epsilon$ under the isomorphism from $u/u^{(4)}$ to $A \oplus B$ is contained in a subgroup $A \oplus C$ where $C$ is of the type described above. Clearly $A \oplus C$ has index 2 in $A \oplus B$. Thus $\epsilon$ is contained in a subgroup $G$ of $u$ of index 2 with $G \supset u^{(4)}$. But then $G$ is open in $u$ and, since $u$ is open in $\hat{F}$, $G$ is open in $\hat{F}$. Denoting by $(\pi)$ the cyclic subgroup of $\hat{F}$ generated by $\pi$, we see that $G(\pi) = \bigcup_{n \in \mathbb{Z}} G\pi^n$ is also open in $\hat{F}$ and $(\hat{F}:G(\pi)) = [u(\pi):G(\pi)] = 2$. Thus $G(\pi)$ is an open subgroup of $\hat{F}$ of index 2 that contains $1, \pi, \epsilon$ and $\pi \epsilon$. Q.E.D.

**THEOREM.** If $V$ is a nondefective quadratic space over the local field $F$ then $O'(V) = \Omega(V)$.

**PROOF.** In light of 1(i), 2(iii) and the results of [3] we may assume that $V$ is quaternary anisotropic. Hence $V = P_1 \perp P_2$ where

$$P_1 \cong \begin{pmatrix} 1 \\ 1 \\ \rho \end{pmatrix} \quad \text{and} \quad P_2 \cong \begin{pmatrix} 1 \\ \pi \\ \pi^{-1} \rho \end{pmatrix}.$$ 

Let $\sigma \in O'(V)$. The universality of $V$ and the results of §1 allow us to assume that $\sigma = \tau_{x_1} \tau_{x_2} \tau_{x_3} \tau_{x_4}$ where $Q(x_i) = 1$. Since $\tau_{x} = \tau_{xz}$, $\alpha \in \hat{F}$, we may assume that $Q(x_i) = \pi$ or $\pi \epsilon_i$, $\epsilon_i \in u$, $1 \leq i \leq 4$. Since $\Pi_{i=1}^{4} Q(x_i) \subseteq \mathbb{F}^4$ we have two cases to consider:
(a) \( Q(x_i) \subseteq u, 1 \leq i \leq 4 \).

In this case \( Q(x_i) \subseteq Q(P_i) \) by Proposition 2(i). Now we apply 1(iv) and 1(i).

(b) Exactly two of the \( Q(x_i), 1 \leq i \leq 4 \), are of the form \( Q(x_i) = \pi \epsilon_i, \epsilon_i \in u \). We may assume that \( Q(x_1) = \epsilon_1, Q(x_2) = \pi \epsilon_2 \) and \( Q(x_4) = \pi \epsilon_4 \) by Proposition 1(iii).

Since \( V \) is universal we may select \( y_1 \) and \( y_2 \) in \( V \) with \( Q(y_1) = \pi \) and \( Q(y_2) = \pi \epsilon_2 \). Let \( \Sigma = \tau_{y_1} \tau_{y_2} \tau_{x_1} \tau_{x_2} \).
Clearly \( \Sigma \in \Omega(V)^\prime \). Moreover, by applying 2(i), 1(i) and 1(iv) to \( P_2 \) we see that \( \Sigma \in \Omega(V) \).

If \( \sigma \Sigma \in \Omega(V) \) we will be done. By applying 1(iii) we see that \( \sigma \Sigma \in \Omega(V) \) if \( \tau_{x_1} \tau_{x_2} \tau_{y_1} \tau_{y_2} \in \Omega(V) \).

By applying 1(iv) to the spaces \( P_1 \) and \( P_2 \) respectively, we see that \( \tau_{x_1} \tau_{x_2} \) and \( \tau_{y_1} \tau_{y_2} \) are in \( \Omega(V)^\prime \). Thus we may assume that \( \epsilon_2 \in u^2 \).

By the above lemma there is a nondefective, binary, anisotropic space \( P \) which represents \( 1, \pi, \pi \epsilon_2 \) and \( \epsilon_2 \). We have \( P \rightarrow V \) by 2(iv). Thus \( V \) contains a nondefective binary space \( B \) which represents \( 1, \pi, \pi \epsilon_2 \) and \( \epsilon_2 \). Applying 1(iv) and 1(i) to \( B \) yields the desired result. Q.E.D.

References


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