ISOMETRIES BETWEEN $B^*$-ALGEBRAS

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In [2], Kadison proves the following theorem:

Let $A$ and $B$ be $C^*$-algebras each of which contains an identity. Then if $T$ is a linear isometry mapping $A$ onto $B$, there exists a $C^*$-isomorphism $\tau$ mapping $A$ onto $B$ (i.e. $\tau$ is a linear isomorphism which preserves self-adjoints and power structure) and a unitary element $v \in B$ such that $T = \tau v$.

In recent years, the theory of numerical range, developed in [3], has provided techniques which have considerably simplified the proofs of certain results in the theory of $B^*$-algebras. The following question, posed by G. Lumer at the North British Functional Analysis Seminar held at Edinburgh in April 1968, is, therefore, natural: Can one prove the above theorem of Kadison using the techniques of the theory of numerical range? Lumer showed that such a proof can be given when the algebras concerned are commutative.

In this paper, we give a simple, intrinsic proof of Kadison's result, using certain elementary notions from the theory of numerical range.

We note that if $A$ is a $B^*$-algebra with identity $1$, the set

$$H = \{ x \in A : \| 1 + i\alpha x \| \leq 1 + o(\alpha), \alpha \in \mathbb{R}, \alpha \to 0 \}$$

coincides with the set of selfadjoint elements of $A$. This is proved by [3, Theorem 21].

In the sequel, $A$ and $B$ are $B^*$-algebras, each containing an identity $1$, and $T$ is a linear isometry mapping $A$ onto $B$. $A_1$ and $H(A)$ denote respectively the closed unit ball and the set of hermitian elements of $A$. $A'$ denotes the space of continuous linear functionals on $A$. $D_A(1)$ is the subset of $A'$ given by $D_A(1) = \{ f \in A' : \| f \| = 1 = f(1) \}$. For $x \in A$, $Sp_A(x)$ denotes the spectrum of $x$ in $A$.

Analogous notations will be used to denote the corresponding sets associated with $B$.

Lemma 1. Let $v$ be an extreme point of $B_1$. Then $v^*v$ is an idempotent.

Proof. The proof is contained in [2, Theorem 1]. It is shown that, if $C$ is the closed subalgebra in $B$ generated by $1$ and $v^*v$, then $v^*v$, regarded as a function on the carrier space of $C$, can assume no values different from $0$ and $1$. The result follows immediately.

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Lemma 2. Let \( u \) be a unitary element of \( A \). Then \( Tu \) is neither a left nor a right divisor of zero in \( B \).

Proof. Let \( u \) be a unitary element of \( A \) and let \( x \in B \) for which \((Tu)x = 0\). Since \( T \) maps \( A \) onto \( B \), \( \exists y \in A \) such that \( x = (Ty)^* \). Hence \((Tu)(Ty)^* = 0 = (Ty)(Tu)^* \). Let \( \alpha \in \mathbb{C} \). Then
\[
\|u + \alpha y\|^2 = \|(Tu + \alpha Ty)^*\|^2 = \|(Tu + \alpha Ty)((Tu)^* + \alpha(Ty)^*)\|
\]
\[
= \|(Tu)(Tu)^* + \alpha \|^2(Ty)(Ty)^*\| \leq \|Tu\|^2 + |\alpha|^2\|Ty\|^2.
\]
This gives \( \|u + \alpha y\| \leq (1 + |\alpha|^2)^{1/2} \), where \( k = \|Ty\|^2 \). Since \( u \) is unitary, \( \|1 + \alpha u^*y\| = \|u + \alpha y\| \leq (1 + |\alpha|^2)^{1/2} \). It follows that as \( \alpha \to 0 \) with \( \alpha \in \mathbb{R} \), we have both
\[
\|1 + \alpha u^*y\| \leq 1 + o(\alpha), \quad \|1 + i\alpha u^*y\| \leq 1 + o(\alpha).
\]
Therefore \( u^*y \in H(A) \cap iH(A) = (0) \).

Since \( u^* \) is regular, \( y = 0 \). Thus \( x = (Ty)^* = 0 \). Hence, \( Tu \) is not a left divisor of zero in \( B \), and it may be similarly shown that \( Tu \) is not a right divisor of zero in \( B \).

It is obvious that Lemma 2 remains true if \( Tu \) is replaced by \((Tu)^*\) in its statement.

Lemma 3. Let \( u \) be a unitary element of \( A \). Then \( Tu \) is a unitary element of \( B \).

Proof. Since \( F \) is a linear isometry of \( A \) onto \( B \), \( T \) maps the extreme points of \( A \) onto the extreme points of \( B \). By [1, Theorem 3], \( u \) is a vertex, and hence an extreme point, of \( A \). Thus \( Tu \) is an extreme point of \( B \).

Let \( p = (Tu)^*Tu \). It follows easily from Lemma 2 that \( p \) is not a divisor of zero.

Now, since by Lemma 1, \( p \) is an idempotent in \( B \), we have \( p(p - 1) = 0 \). Hence \( p = 1 \), i.e. \((Tu)^*Tu = 1\).

Now, if \( y \in B \) is an extreme point of \( B \), \( y^* \) is also an extreme point of \( B \). Hence \((Tu)^* \) is an extreme point of \( B \). Applying the above argument to \((Tu)^* \) instead of to \( Tu \), it is clear that \((Tu)(Tu)^* = 1\). Hence \( Tu \) is a unitary point of \( B \).

Theorem. Let \( A \) and \( B \) be \( B^*-\)algebras each containing an identity \( 1 \). Then if \( T \) is a linear isometry mapping \( A \) onto \( B \), there exists a unitary element \( v \) of \( B \) and a \( C^*-\)isomorphism \( \tau \) of \( A \) onto \( B \) such that \( T = vr \).

Proof. Let \( T \) be a linear isometry of \( A \) onto \( B \). By Lemma 3, \( T1 \) is a unitary element of \( B \). Let \( v = T(1) \), and define the mapping \( \tau \) of \( A \) into \( B \) by \( \tau = v^*T \). \( \tau \) is clearly linear, and maps \( A \) onto \( B \). Further,
since \( v^* \) is unitary, \( \| \tau(x) \| = \| v^* T(x) \| = \| Tx \| = \| x \| \) for \( x \) in \( A \). Thus \( \tau \) is an isometry, and \( \tau(1) = v^* v = 1 \).

Since \( T = v \tau \), if we can show that \( \tau \) is a \( C^* \)-isomorphism, the theorem will be proved. Let \( h \in H(A) \). Then

\[
\| 1 + i\alpha \tau(h) \| = \| 1 + i\alpha h \| \leq 1 + o(\alpha) \quad (\alpha \in \mathbb{R}, \alpha \to 0).
\]

Thus \( \tau(h) \in H(B) \) and \( \tau \) is a \( * \)-mapping.

Finally, we must prove that \( \tau(x^2) = [\tau(x)]^2 \) (\( x \in A \)). Let \( h \in H(A) \), \( \alpha \in \mathbb{R} \). Then \( e^{i\alpha}h \) is a unitary element of \( A \). By Lemma 3 applied to \( \tau \), \( \tau(e^{i\alpha}h) \) is a unitary element of \( B \), i.e. \( \tau(e^{i\alpha}) \tau(e^{-i\alpha}) = 1 \).

Thus \( [1 + i\alpha \tau(h) - \alpha^2 \tau(h^2)/2][1 - i\alpha \tau(h) - \alpha^2 \tau(h^2)/2] = 1 + O(\alpha^2) \) as \( \alpha \to 0 \), using the fact that \( \tau \) is continuous and \( \tau(1) = 1 \).

Hence \( 1 + \alpha^2 \left( [\tau(h)]^2 - \tau(h^2) \right) = 1 + O(\alpha^2) \) as \( \alpha \to 0 \); i.e. \( [\tau(h)]^2 - \tau(h^2) = O(\alpha) \) as \( \alpha \to 0 \); \( [\tau(h)]^2 = \tau(h^2) \).

Now let \( x \in A \), \( x = h + ik \), where \( h, k \in H(A) \). Since \( [\tau(h+k)]^2 = \tau(h+k)^2 \), we have

\[
\tau(hk + kh) = \tau(h) \tau(k) + \tau(k) \tau(h).
\]

Hence \( \tau(x^2) = \tau(h^2 - k^2 + i(hk + kh)) = [\tau(x)]^2 \) (\( x \in A \)). Thus \( \tau \) is a \( C^* \)-isomorphism.

Note. The converse of the above theorem was also proved by Kadison [2], i.e. if \( \tau \) is a \( C^* \)-isomorphism mapping \( A \) onto \( B \) and \( v \) is a unitary element in \( B \), then \( T = v \tau \) is a linear isometry of \( A \) onto \( B \). This result is an easy consequence of [4, Corollary 1].

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References


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