ISOMETRIES BETWEEN $B^*$-ALGEBRAS

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In [2], Kadison proves the following theorem:

Let $A$ and $B$ be $C^*$-algebras each of which contains an identity. Then
if $T$ is a linear isometry mapping $A$ onto $B$, there exists a $C^*$-isomor-
phism $\tau$ mapping $A$ onto $B$ (i.e. $\tau$ is a linear isomorphism which pre-
serves selfadjoints and power structure) and a unitary element $v \in B$
such that $T = \tau v$.

In recent years, the theory of numerical range, developed in [3],
has provided techniques which have considerably simplified the proofs
of certain results in the theory of $B^*$-algebras. The following question,
posed by G. Lumer at the North British Functional Analysis Seminar
held at Edinburgh in April 1968, is, therefore, natural: Can one prove
the above theorem of Kadison using the techniques of the theory of
numerical range? Lumer showed that such a proof can be given when
the algebras concerned are commutative.

In this paper, we give a simple, intrinsic proof of Kadison's result,
using certain elementary notions from the theory of numerical range.

We note that if $A$ is a $B^*$-algebra with identity 1, the set

$$H = \{ x \in A : \| 1 + i \alpha x \| \leq 1 + o(\alpha), \alpha \in \mathbb{R}, \alpha \to 0 \}$$

coincides with the set of selfadjoint elements of $A$. This is proved by
[3, Theorem 21].

In the sequel, $A$ and $B$ are $B^*$-algebras, each containing an identity
1, and $T$ is a linear isometry mapping $A$ onto $B$. $A_1$ and $H(A)$ denote
respectively the closed unit ball and the set of hermitian elements of
$A$. $A'$ denotes the space of continuous linear functionals on $A$. $DA(1)$
is the subset of $A'$ given by $DA(1) = \{ f \in A' : \| f \| = 1 = f(1) \}$. For
$x \in A$, $\text{Sp}_A(x)$ denotes the spectrum of $x$ in $A$.

Analogous notations will be used to denote the corresponding sets
associated with $B$.

**Lemma 1.** Let $v$ be an extreme point of $B_1$. Then $v^*v$ is an idempotent.

**Proof.** The proof is contained in [2, Theorem 1]. It is shown that,
if $C$ is the closed subalgebra in $B$ generated by 1 and $v^*v$, then $v^*v$, regarded as a function on the carrier space of $C$, can assume no values
different from 0 and 1. The result follows immediately.

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**Lemma 2.** Let \( u \) be a unitary element of \( A \). Then \( Tu \) is neither a left nor a right divisor of zero in \( B \).

**Proof.** Let \( u \) be a unitary element of \( A \) and let \( x \in B \) for which \( (Tu)x = 0 \). Since \( T \) maps \( A \) onto \( B \), \( \exists y \in A \) such that \( x = (Ty)^* \). Hence \( (Tu)(Ty)^* = 0 = (Ty)(Tu)^* \). Let \( \alpha \in \mathbb{C} \). Then

\[
\|u + \alpha y\|^2 = \|Tu + \alpha Ty\|^2 = \|(Tu + \alpha Ty)((Tu)^* + \alpha(Ty)^*)\|
\]

\[
= \|(Tu)(Tu)^* + |\alpha|^2(Ty)(Ty)^*\| \leq \|Tu\|^2 + |\alpha|^2\|Ty\|^2.
\]

This gives \( \|u + \alpha y\| \leq (1 + |\alpha|^2)^{1/2} \), where \( k = \|Ty\|^2 \). Since \( u \) is unitary, \( \|1 + \alpha u^* y\| = \|u + \alpha y\| \leq (1 + |\alpha|^2)^{1/2} \). It follows that as \( \alpha \to 0 \) with \( \alpha \in \mathbb{R} \), we have both

\[
\|1 + \alpha u^* y\| \leq 1 + o(\alpha), \quad \|1 + i\alpha u^* y\| \leq 1 + o(\alpha).
\]

Therefore \( u^* y \in H(A) \cap iH(A) = (0) \).

Since \( u^* \) is regular, \( y = 0 \). Thus \( x = (Ty)^* = 0 \). Hence, \( Tu \) is not a left divisor of zero in \( B \), and it may be similarly shown that \( Tu \) is not a right divisor of zero in \( B \).

It is obvious that Lemma 2 remains true if \( Tu \) is replaced by \( (Tu)^* \) in its statement.

**Lemma 3.** Let \( u \) be a unitary element of \( A \). Then \( Tu \) is a unitary element of \( B \).

**Proof.** Since \( T \) is a linear isometry of \( A \) onto \( B \), \( T \) maps the extreme points of \( A \) onto the extreme points of \( B \). By [1, Theorem 3], \( u \) is a vertex, and hence an extreme point, of \( A \). Thus \( Tu \) is an extreme point of \( B \).

Let \( p = (Tu)^*Tu \). It follows easily from Lemma 2 that \( p \) is not a divisor of zero.

Now, since by Lemma 1, \( p \) is an idempotent in \( B \), we have \( p(p - 1) = 0 \). Hence \( p = 1 \), i.e. \( (Tu)^*Tu = 1 \).

Now, if \( y \in B \) is an extreme point of \( B \), \( y^* \) is also an extreme point of \( B \). Hence \( (Tu)^* \) is an extreme point of \( B \). Applying the above argument to \( (Tu)^* \) instead of to \( Tu \), it is clear that \( (Tu)(Tu)^* = 1 \). Hence \( Tu \) is a unitary point of \( B \).

**Theorem.** Let \( A \) and \( B \) be \( B^* \)-algebras each containing an identity \( 1 \). Then if \( T \) is a linear isometry mapping \( A \) onto \( B \), there exists a unitary element \( v \) of \( B \) and a \( C^* \)-isomorphism \( \tau \) of \( A \) onto \( B \) such that \( T = vr \).

**Proof.** Let \( T \) be a linear isometry of \( A \) onto \( B \). By Lemma 3, \( T1 \) is a unitary element of \( B \). Let \( v = T(1) \), and define the mapping \( \tau \) of \( A \) into \( B \) by \( \tau = v^* T \). \( \tau \) is clearly linear, and maps \( A \) onto \( B \). Further,
since $v^*$ is unitary, $\|r(x)\| = \|v^*T(x)\| = \|Tx\| = \|x\|$ for $x$ in $A$. Thus $\tau$ is an isometry, and $\tau(1) = v^*v = 1$.

Since $T = vr$, if we can show that $\tau$ is a $C^*$-isomorphism, the theorem will be proved. Let $h \in H(A)$. Then
\[
\|1 + i\alpha \tau(h)\| = \|1 + i\alpha h\| \leq 1 + o(\alpha) \quad (\alpha \in \mathbb{R}, \alpha \to 0).
\]

Thus $\tau(h) \in H(B)$ and $\tau$ is a $\ast$-mapping.

Finally, we must prove that $\tau(x^2) = [\tau(x)]^2$ ($x \in A$). Let $h \in H(A)$, $\alpha \in \mathbb{R}$. Then $e^{i\alpha}$ is a unitary element of $A$. By Lemma 3 applied to $\tau$, $\tau(e^{i\alpha})$ is a unitary element of $B$, i.e. $\tau(e^{i\alpha})\tau(e^{-i\alpha}) = 1$.

Thus $[1 + i\alpha \tau(h) - \alpha^2\tau(h^2)/2][1 - i\alpha \tau(h) - \alpha^2\tau(h^2)/2] = 1 + O(\alpha^2)$ as $\alpha \to 0$, using the fact that $\tau$ is continuous and $\tau(1) = 1$.

Hence $1 + \alpha^2[\tau(h)^2 - \tau(h^2)] = 1 + O(\alpha^2)$ as $\alpha \to 0$; i.e. $[\tau(h)]^2 - \tau(h^2) = O(\alpha)$ as $\alpha \to 0$; $[\tau(h)]^2 = \tau(h^2)$.

Now let $x \in A$, $x = h + ik$, where $h$, $k \in H(A)$. Since $[\tau(h + k)]^2 = \tau[(h + k)^2]$, we have
\[
\tau(hk + kh) = \tau(h)\tau(k) + \tau(k)\tau(h).
\]

Hence $\tau(x^2) = \tau(h^2 - k^2 + i(hk + kh)) = [\tau(x)]^2$ ($x \in A$). Thus $\tau$ is a $C^*$-isomorphism.

Note. The converse of the above theorem was also proved by Kadison [2], i.e. if $\tau$ is a $C^*$-isomorphism mapping $A$ onto $B$ and $v$ is a unitary element in $B$, then $T = vr$ is a linear isometry of $A$ onto $B$. This result is an easy consequence of [4, Corollary 1].

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References


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