THE SPECTRUM OF A LINEAR OPERATOR UNDER PERTURBATION BY CERTAIN COMPACT OPERATORS

KENNETH K. WARNER

Let $T$ be a bounded linear operator on a Banach space $X$. A subset of the spectrum of $T$ which is invariant under certain compact perturbation of $T$ is studied. It consists of the spectrum of $T$ with finite-dimensional poles deleted. In the case of a bounded operator, it coincides with the essential spectrum as defined by F. E. Browder [1]. It is characterized as a set considered by Caradus [2]. A formula of the spectral radius type is proved. Furthermore, a spectral mapping theorem is valid.

The notation is that of Taylor [5]. Let $R(T)$ denote the range of $T$ and $N(T)$ the nullspace of $T$, i.e., $N(T) = \{x :Tx = 0\}$. The dimension of $N(T)$, $n(T)$, is called the nullity of $T$ and the codimension of $R(T)$, $d(T)$, the defect of $T$. Suppose for some integer $k$, $N(T^k) = N(T^{k+1})$; then the ascent, $\alpha(T)$, is defined as the smallest value of $k$ for which this is true. The smallest integer for which $R(T^k) = R(T^{k+1})$ is called the descent of $T$ and is denoted by $\delta(T)$. For the operator $\lambda - T$, $n(\lambda - T)$ is abbreviated to $n(\lambda)$, etc. $B(X)$ will denote the bounded linear operators, $C(X)$ the compact linear operators. $A \perp B$ means $AB = BA = 0$. Let $[T] \in B(X)/C(X)$; then $\sigma([T])$ denotes the spectrum of $[T]$ as an element of that Banach algebra. For a linear operator $T$, let $P(T) = \{C \in C(X) : T - C \perp C\}$ and $Q(T) = \{D \in C(X) : DT = TD\}$. The object of this paper is to study the sets

$$\sigma_{P(T)} = \bigcap_{C \in P(T)} \sigma(T - C), \quad \sigma_{Q(T)} = \bigcap_{D \in Q(T)} \sigma(T - D).$$

The complement of $\sigma_{P(T)}$ will be denoted by $\rho_{P(T)}$, and the complement of $\sigma_{Q(T)}$ will be denoted by $\rho_{Q(T)}$. When no confusion will arise, the $T$ will be suppressed.

**Lemma 1.** $\sigma_{P}$ is a closed set, and $\sigma([T]) \subseteq \sigma_{P} \subseteq \sigma(T)$.

**Proof.** $\sigma_{P}$ is closed because it is the intersection of closed sets. Since $0 \in P$, then $\sigma_{P} \subseteq \sigma(T - 0) = \sigma(T)$.

Let $\lambda \in \rho_{P}$; then there is a $C \in P$ such that $\lambda \in \rho(T - C)$. Thus, $R_{\lambda}(\lambda - T + C) = I$, where $R_{\lambda} = (\lambda - T + C)^{-1}$, the resolvent operator. Then $[R_{\lambda}][\lambda - T] = [\lambda - T][R_{\lambda}] = [I]$. This implies that $\lambda \in \rho([T])$, and $\rho_{P} \subseteq \rho([T])$. Hence, $\sigma([T]) \subseteq \sigma_{P}$.

Received by the editors March 28, 1967 and, in revised form, November 14, 1968.

667
Lemma 2. Let $T \in B(X)$. Suppose $\lambda_0 \neq 0$ is an isolated point of $\sigma(T)$. Let $E_0$ be the spectral projection associated with $\lambda_0$. Then $T - TE_0$ and $\lambda_0 \notin \sigma(T - TE_0)$.

Proof. The operational calculus for $T$ (see [4]) implies that $TE_0 = E_0T$ and $(I - E_0)E_0 = E_0(I - E_0) = 0$. These statements give $T - TE_0$.

Let $f(\lambda) = \lambda$ on a neighborhood of $\sigma(T) \sim \{\lambda_0\}$ and $f(\lambda) = 0$ on a neighborhood of $\{\lambda_0\}$. Then $f \in \mathcal{H}_\omega(T)$, and $f(T) = T - TE_0$. The spectral mapping theorem implies $\lambda_0 \notin \sigma(T - TE_0)$.

Theorem 1. (a) $\rho_{F(T)} \sim \{0\} = \{\lambda : n(\lambda) = d(\lambda) \text{ and } \delta(\lambda) = \alpha(\lambda)\}$

(b) $\rho_{Q(T)} = \{\lambda : n(\lambda) = d(\lambda) \text{ and } \delta(\lambda) = \alpha(\lambda)\}$

Proof. Let $\lambda \in \rho_Q$; then there is a $D \in Q$ such that $\lambda \in \rho(T - D)$. We can write $\lambda - T = (\lambda - (T - D)) + (D - D)$. Let $U = \lambda - (T - D)$. Then $U$ has the properties that it has a bounded inverse, $(\lambda - T - U)$ is compact, and $(\lambda - T - U)U = U(\lambda - T) = TD = DT$ (since $TD = DT$). Thus, Theorem 6.3 of Yood [6] implies that $n(\lambda) = d(\lambda)$ and $\alpha(\lambda) = \delta(\lambda)$. Also, $\rho_F \subset \rho_Q$.

Let $\lambda \in \sigma(T)$ such that $n(\lambda) = d(\lambda)$ and $\alpha(\lambda) = \delta(\lambda)$. Now Theorem 9.4 of Taylor [5] shows that $\lambda$ is an isolated point of $\sigma(T)$. Then by Corollary 9.3 of Taylor [5], we conclude that $E_\lambda$, the associated spectral projection, is a finite-dimensional operator. Thus, $TE_\lambda$ is compact. If $\lambda \neq 0$, then Lemma 2 implies $T - TE_\lambda$.

To prove (b), it suffices from the above to consider $\lambda = 0$. For $\mu \neq 0$, $\mu \in \mathcal{F}$ has a finite-dimensional pole at $\mu$, and the associated spectral projection $E_\mu = E_0$, by Theorem 5.71D of Taylor [4]. By Lemma 2, $\mu \in \sigma(T_\mu - T_\mu E_0)$. Hence $(\mu - (T_\mu - T_\mu E_0))^{-1} = (T + T_\mu E_0)^{-1}$ exists, and $T_\mu E_0 \in Q$. This proves (b).

Caradus [2] defined the Riesz region, $\mathcal{R}_T$, of $T$ to be $\{\lambda : n(\lambda)$ and $\delta(\lambda)$ are finite $\}$; the Fredholm region, $\mathcal{FR}_T$, to be $\{\lambda : n(\lambda)$ and $d(\lambda)$ are finite $\}$.

Corollary 1. $\rho_{Q(T)} = \mathcal{R}_T \cap \mathcal{FR}_T$. Hence $\mathcal{R}_T \cap \mathcal{FR}_T$ is open.

Proof. Theorem 6.1 of Yood [6] or Lemma 2 of Caradus [2] imply that $\mathcal{R}_T \cap \mathcal{FR}_T = \{n(\lambda) = d(\lambda) \text{ and } \alpha(\lambda) = \delta(\lambda)\}$.

Theorem 1 completes the proof.

Corollary 2. $\lambda \in \sigma_Q(T)$ if and only if either $\lambda$ is a limit point of $\sigma(T)$, or $\lambda$ is an isolated point whose associated spectral projection is infinite dimensional.
Theorem 2.
\[ r = \lim_{n} \left\{ \inf_{C \in P} \left\| T^{n} - C^{n} \right\| \right\}^{1/n}. \]

Proof. Since \( T - C \perp C \), we have by induction \( (T - C)^{n} = T^{n} - C^{n} \).

Let \( r(A) \) be the spectral radius of \( A \in B(X) \). It is well known that \( r(A_{n}) = (r(A))^{n} \) and \( \left\| A^{n} \right\| \geq (r(A))^{n} \). Hence, for \( C \in P \)

\[ \left\| (T - C)^{n} \right\| \geq (r(T - C))^{n} \geq r^{n}. \]

For each \( n \),

\[ \left\{ \inf_{C \in P} \left\| T^{n} - C^{n} \right\| \right\}^{1/n} \geq r. \]

Let \( a > r \). Pick \( p \) such that \( a > p > r \). Then if \( \left| \lambda \right| > p \), we have \( n(\lambda) = d(\lambda) \) and \( \alpha(\lambda) = \delta(\lambda) \). If \( \lambda \in \sigma(T) \) and \( \left| \lambda \right| > p \), then Theorem 9.4 of Taylor [5] implies that \( \lambda \) is an isolated point of \( \sigma(T) \), and Corollary 9.3 of Taylor [5] that the associated spectral projection is a finite dimensional operator.

There can only be a finite number of such points \( \lambda \in \sigma(T) \) and \( \left| \lambda \right| > p \) (for Theorem 9.4 of Taylor [5] would imply that a limit point of such points would be isolated). Denote these points by \( \{ \lambda_{i} \}_{n}^{n} \). Let \( E_{i} \) be the finite-dimensional projection associated with \( \lambda_{i} \). Then the operational calculus for \( T \) gives \( C = T( \sum_{i} E_{i} ) \in P \), and the spectral mapping theorem that \( \lambda_{i} \in \sigma(T - C) \) for \( i = 1, \ldots, n \). Hence, \( p \geq r(T - C) \).

Thus, by the spectral radius theorem there is an \( N \) such that \( a > \left\| (T - C)^{n} \right\|^{1/n} \geq r \) for \( n \geq N \). Thus \( a^n > \left\| (T - C)^{n} \right\| \geq r^n \). But \( \left\| (T - C)^{n} \right\| \geq \left\| T^n - C^n \right\| \geq \inf_{C \in P} \left\| T^n - C^n \right\| \). Hence,

\[ a^n > \inf_{C \in P} \left\| T^n - C^n \right\| \geq r^n, \quad \text{or} \quad a > \left\{ \inf_{C \in P} \left\| T^n - C^n \right\| \right\}^{1/n} \geq r, \]

which completes the proof.

The norm in the Banach algebra \( B(X)/C(X) \) is given by \( K(T) = \inf_{C \in C(X)} \left\| T - C \right\| \) where \( C \in C(X) \). The next theorem shows the spectral radius of an element of \( B(X)/C(X) \) is \( r \).
Theorem 3. For any $T \in B(X)$,
\[ r = \lim_{n \to \infty} |K(T^n)|^{1/n}. \]

Proof. Let $s = \lim_{n \to \infty} |K(T^n)|^{1/n}$. Then $s$ is the spectral radius of the element $[T]$ in $B(X)/C(X)$. Since $G = \{ \lambda : |\lambda| > s \}$ is an open connected set, Theorem 3.3 and its corollary of Gohberg and Krein [3] imply that $\sigma(T) \cap G$ consists of isolated points of $\sigma(T)$ such that $n(\lambda) < \infty$. Hence, Corollary 9.3 of Taylor [5] implies that the spectral projections associated with each of these is finite dimensional. Let $l$ be arbitrary and $l > s$. Then there are only a finite number of points $\lambda \in \sigma(T)$ and $|\lambda| \geq l$. Let $\sigma$ denote the spectral set consisting of these points. Let $E_\sigma$ be the spectral projection associated with $\sigma$. Then, as before, $T - TE_\sigma$ has spectrum inside the circle $|\lambda| = l$. $TE_\sigma$ is a finite dimensional operator. Thus $l > r$. Lemma 1 implies that $r \geq s$. Hence $r = s$.

The operational calculus of an operator $T$ allows one to assign an operator $f(T)$ for every function $f$ analytic on a neighborhood of $\sigma(T)$ (see Taylor [4]). The following type of "spectral mapping" theorem is valid.

Theorem 4. Let $f$ be analytic on an open set containing $\sigma(T)$. Suppose for each $\lambda_0$ that $\{ \lambda : f(\lambda) = f(\lambda_0) \}$ is finite. Then $f(\sigma_q(T)) = \sigma_q(f(T))$.

Proof. Suppose $\lambda_0 \in \sigma_q(T)$. Since the spectral mapping theorem implies that $f(\sigma(T)) = \sigma(f(T))$, $f(\lambda_0)$ is either a limit point of $\sigma(f(T))$ or an isolated point. If $f(\lambda_0)$ is a limit point, Corollary 2 implies that $f(\lambda_0) \in \sigma_q(f(T))$. If $f(\lambda_0)$ is isolated, then Theorem 5.71D of Taylor [4] implies that $\sigma = \{ \lambda : f(\lambda) = f(\lambda_0) \} \cap \sigma(T)$ is a finite spectral set of $T$, and the spectral projection associated with $\sigma$ and $T$, $E_\sigma(T)$, equals that associated with $f(\lambda_0)$ and $F_f(\lambda_0)(f(T))$, i.e. $E_\sigma(T) = F_f(\lambda_0)(f(T))$. Since $\sigma$ is a finite spectral set, this implies that $\lambda_0$ is an isolated point. Corollary 2 implies that $E_\lambda$ is infinite dimensional. Hence $F_f(\lambda_0)$ is infinite dimensional. Thus $f(\lambda_0) \in \sigma_q(f(T))$, or $f(\sigma_q(T)) \subseteq \sigma_q(f(T))$.

Suppose that $\mu \in \sigma_q(f(T))$. If $\mu$ is a limit point of $\sigma(f(T))$, then since $\sigma(f(T)) = \sigma(f(T))$, there is a limit point $\lambda$ of $\sigma(T)$ such that $f(\lambda) = \mu$. Corollary 2 implies that $\lambda \in \sigma_q(T)$. If $\mu$ is isolated, then, as before, $\sigma = \{ \lambda : f(\lambda) = \mu \} \cap \sigma(T)$ is a nonempty finite spectral set such that $E_\sigma(T) = F_\sigma(f(T))$. Since points of $\sigma$ are isolated, $E_\sigma$ is the finite sum of the spectral projections associated with the points of $\sigma$. Since $F_\mu$ is infinite dimensional, one of these projections must be infinite dimensional. Thus there is a $\lambda \in \sigma$ such that $f(\lambda) = \mu$ and $\lambda \in \sigma_q(T)$. Thus, $f(\sigma_q(T)) = \sigma_q(f(T))$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Remark. The above theorems hold if $P(T)$ and $Q(T)$ are replaced with finite-dimensional operators that satisfy the defining conditions for these sets.

References