THE SPECTRUM OF A LINEAR OPERATOR UNDER PERTURBATION BY CERTAIN COMPACT OPERATORS

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Let $T$ be a bounded linear operator on a Banach space $X$. A sub-
set of the spectrum of $T$ which is invariant under certain compact
perturbation of $T$ is studied. It consists of the spectrum of $T$ with
finite-dimensional poles deleted. In the case of a bounded operator,
it coincides with the essential spectrum as defined by F. E. Browder
[1]. It is characterized as a set considered by Caradus [2]. A formula
of the spectral radius type is proved. Furthermore, a spectral map-
ing theorem is valid.

The notation is that of Taylor [5]. Let $R(T)$ denote the range of $T$
and $N(T)$ the nullspace of $T$, i.e., $N(T) = \{x: Tx = 0\}$. The dimension
of $N(T)$, $n(T)$, is called the nullity of $T$ and the codimension of $R(T)$,
d$(T)$, the defect of $T$. Suppose for some integer $k$, $N(T^k) = N(T^{k+1})$;
then the ascent, $a(T)$, is defined as the smallest value of $k$ for which
this is true. The smallest integer for which $R(T^k) = R(T^{k+1})$ is
called the descent of $T$ and is denoted by $b(T)$. For the operator
$\lambda - T$, $n(\lambda - T)$ is abbreviated to $n(\lambda)$, etc. $B(X)$ will denote the
bounded linear operators, $C(X)$ the compact linear operators. $A \supset B$
means $AB = BA = 0$. Let $[T] \in B(X)/C(X)$; then $\sigma([T])$ denotes the
spectrum of $[T]$ as an element of that Banach algebra. For a linear
operator $T$, let $P(T) = \{C \in C(X): T - C \in \mathbb{C}\}$ and $Q(T) = \{D \in C(X):$
$DT = TD\}$. The object of this paper is to study the sets

$$\sigma_P(T) = \bigcap_{C \in P(T)} \sigma(T - C), \quad \sigma_Q(T) = \bigcap_{D \in Q(T)} \sigma(T - D).$$

The complement of $\sigma_P(T)$ will be denoted by $\rho_P(T)$, and the comple-
ment of $\sigma_Q(T)$ will be denoted by $\rho_Q(T)$. When no confusion will arise,
the $T$ will be suppressed.

**Lemma 1.** $\sigma_P$ is a closed set, and $\sigma([T]) \subseteq \sigma_P \subseteq \sigma(T)$.

**Proof.** $\sigma_P$ is closed because it is the intersection of closed sets.
Since $0 \in P$, then $\sigma_P \subseteq \sigma(T - 0) = \sigma(T)$.

Let $\lambda \in \rho_P$; then there is a $C \in P$ such that $\lambda \in \rho(T - C)$. Thus,
$R_\lambda(\lambda - T + C) = I$, where $R_\lambda = (\lambda - T + C)^{-1}$, the resolvent operator.
Then $[R_\lambda][\lambda - T] = [\lambda - T][R_\lambda] = [I]$. This implies that $\lambda \in \rho([T])$,
and $\rho_P \subseteq P([T])$. Hence, $\sigma([T]) \subseteq \sigma_P$.

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Lemma 2. Let \( T \in B(X) \). Suppose \( \lambda_0 \neq 0 \) is an isolated point of \( \sigma(T) \). Let \( E_0 \) be the spectral projection associated with \( \lambda_0 \). Then \( T - TE_0 \perp TE_0 \) and \( \lambda_0 \in \sigma(T - TE_0) \).

Proof. The operational calculus for \( T \) (see [4]) implies that \( TE_0 = E_0 T \) and \( (I - E_0)E_0 = E_0(I - E_0) = 0 \). These statements give \( T - TE_0 \perp TE_0 \).

Let \( f(\lambda) = \lambda \) on a neighborhood of \( \sigma(T) \sim \{ \lambda_0 \} \) and \( f(\lambda) = 0 \) on a neighborhood of \( \{ \lambda_0 \} \). Then \( f \in \mathfrak{A}_0(T) \), and \( f(T) = T - TE_0 \). The spectral mapping theorem implies \( \lambda_0 \in \sigma(T - TE_0) \).

Theorem 1. (a) \( \rho_{F(T)} \sim \{ 0 \} = \{ \lambda : n(\lambda) = d(\lambda) \text{ and } \delta(\lambda) = \alpha(\lambda) \} \sim \{ 0 \} \); (b) \( \rho_{O(T)} = \{ \lambda : n(\lambda) = d(\lambda) \text{ and } \delta(\lambda) = \alpha(\lambda) \} \).

Proof. Let \( \lambda \in \rho_0 \); then there is a \( D \in Q \) such that \( \lambda \in \rho(T - D) \). We can write \( \lambda - T = (\lambda - (T - D)) + (-D) \). Let \( U = \lambda - (T - D) \). Then \( U \) has the properties that it has a bounded inverse, \( (\lambda - T) - U \) is compact, and \( (\lambda - T)U = U(\lambda - T) \) (since \( TD = DT \)). Thus, Theorem 6.3 of Yood [6] implies that \( n(\lambda) = d(\lambda) \) and \( \alpha(\lambda) = \delta(\lambda) \). Also, \( \rho_F \subset \rho_0 \).

Let \( \lambda \in \sigma(T) \) such that \( n(\lambda) = d(\lambda) \) and \( \delta(\lambda) = \alpha(\lambda) \). Now Theorem 9.4 of Taylor [5] shows that \( \lambda \) is an isolated point of \( \sigma(T) \). Then by Corollary 9.3 of Taylor [5], we conclude that \( E_\lambda \), the associated spectral projection, is a finite-dimensional operator. Thus, \( TE_\lambda \) is compact. If \( \lambda \neq 0 \), then Lemma 2 implies \( T - TE_\lambda \perp TE_\lambda \) and \( \lambda \in \sigma(T - TE_\lambda) \). Hence \( \lambda \in \rho_F \sim \{ 0 \} \). Thus we have proved (a).

To prove (b), it suffices from the above to consider \( \lambda = 0 \). For \( \mu \neq 0 \), \( T_\mu = \mu - T \) has a finite-dimensional pole at \( \mu \), and the associated spectral projection \( E_\mu = E_0 \), by Theorem 5.71D of Taylor [4]. By Lemma 2, \( \mu \in \sigma(T_\mu - T_\mu E_0) \). Hence \( (\mu - (T_\mu - T_\mu E_0))^{-1} = (T_\mu + T_\mu E_0)^{-1} \) exists, and \( -T_\mu E_0 \in Q \). This proves (b).

Caradus [2] defined the Riesz region, \( \mathfrak{R}_T \), of \( T \) to be \( \{ \lambda : n(\lambda) \text{ and } \delta(\lambda) \text{ are finite} \} \); the Fredholm region, \( \mathfrak{F}_T \), to be \( \{ \lambda : n(\lambda) \text{ and } d(\lambda) \text{ are finite} \} \).

Corollary 1. \( \rho_{O(T)} = \mathfrak{R}_T \cap \mathfrak{F}_T \). Hence \( \mathfrak{R}_T \cap \mathfrak{F}_T \) is open.

Proof. Theorem 6.1 of Yood [6] or Lemma 2 of Caradus [2] imply that \( \mathfrak{R}_T \cap \mathfrak{F}_T = \{ n(\lambda) = d(\lambda) \text{ and } \alpha(\lambda) = \delta(\lambda) \} \).

Theorem 1 completes the proof.

Corollary 2. \( \lambda \in \sigma_0(T) \) if and only if either \( \lambda \) is a limit point of \( \sigma(T) \), or \( \lambda \) is an isolated point whose associated spectral projection is infinite dimensional.
Proof. Theorem 1, and Theorem 9.3 and Corollary 9.3 of Taylor [5], imply that the points of $\rho_0 \cap \sigma(T)$ are isolated points whose spectral projections are finite-dimensional operators.

Let $r = \sup |\lambda|$ for $\lambda \in \sigma_P(T)$. Then the following spectral radius type theorem is valid.

Theorem 2.

\[ r = \lim_{n \to \infty} \left\{ \inf_{c \in P} \| T^n - C^n \| \right\}^{1/n}. \]

Proof. Since $T - C \perp C$, we have by induction $(T - C)^n = T^n - C^n$. Let $r(A)$ be the spectral radius of $A \in B(X)$. It is well known that $r(A^n) = (r(A))^n$ and $\| A^n \| \geq (r(A))^n$. Hence, for $C \in P$

\[ \| (T - C)^n \| \geq (r(T - C))^n \geq r^n. \]

For each $n$,

\[ \left\{ \inf_{c \in P} \| T^n - C^n \| \right\}^{1/n} \geq r. \]

Let $a > r$. Pick $p$ such that $a > p > r$. Then if $|\lambda| > p$, we have $n(\lambda) = d(\lambda)$ and $\alpha(\lambda) = \delta(\lambda)$. If $\lambda \in \sigma(T)$ and $|\lambda| > p$, then Theorem 9.4 of Taylor [5] implies that $\lambda$ is an isolated point of $\sigma(T)$, and Corollary 9.3 of Taylor [5] that the associated spectral projection is a finite dimensional operator.

There can only be a finite number of such points $\lambda \in \sigma(T)$ and $|\lambda| > p$ (for Theorem 9.4 of Taylor [5] would imply that a limit point of such points would be isolated). Denote these points by $\{ \lambda_i \}$.

Let $E_i$ be the finite-dimensional projection associated with $\lambda_i$. Then the operational calculus for $T$ gives $C = T(\sum_i E_i) \in P$, and the spectral mapping theorem that $\lambda_i \in \sigma(T - C)$ for $i = 1, \ldots, n$. Hence, $\rho \geq r(T - C)$.

Thus, by the spectral radius theorem there is an $N$ such that $a > \| (T - C)^n \|^{1/n} \geq r$ for $n \geq N$. Thus $a^n > \| (T - C)^n \| \geq r^n$. But $\| (T - C)^n \| \geq \| T^n - C^n \| \geq \inf_{c \in P} \| T^n - C^n \|$. Hence,

\[ a^n > \inf_{c \in P} \| T^n - C^n \| \geq r^n, \quad \text{or} \quad a > \left\{ \inf_{c \in P} \| T^n - C^n \| \right\}^{1/n} \geq r, \]

which completes the proof.

The norm in the Banach algebra $B(X)/C(X)$ is given by $K(T) = \inf_{C \in C} \| T - C \|$ where $C \subseteq C(X)$. The next theorem shows the spectral radius of an element of $B(X)/C(X)$ is $r$. 

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Theorem 3. For any $T \in B(X)$,$$
r = \lim_{n \to \infty} (K(T^n))^{1/n}.
$$

Proof. Let $s = \lim_{n \to \infty} (K(T^n))^{1/n}$. Then $s$ is the spectral radius of the element $T$ in $B(X)/C(X)$. Since $G = \{ \lambda : |\lambda| > s \}$ is an open connected set, Theorem 3.3 and its corollary of Gohberg and Krein [3] imply that $\sigma(T) \cap G$ consists of isolated points of $\sigma(T)$ such that $n(\lambda) < \infty$. Hence, Corollary 9.3 of Taylor [5] implies that the spectral projections associated with each of these is finite dimensional. Let $l$ be arbitrary and $l > s$. Then there are only a finite number of points $\lambda \in \sigma(T)$ and $|\lambda| \geq l$. Let $\sigma$ denote the spectral set consisting of these points. Let $E_\sigma$ be the spectral projection associated with $\sigma$. Then, as before, $T - TE_\sigma$ has spectrum inside the circle $|\lambda| = l$. $TE_\sigma$ is a finite dimensional operator. Thus $l > r$. Lemma 1 implies that $r \geq s$. Hence $r = s$.

The operational calculus of an operator $T$ allows one to assign an operator $f(T)$ for every function $f$ analytic on a neighborhood of $\sigma(T)$ (see Taylor [4]). The following type of "spectral mapping" theorem is valid.

Theorem 4. Let $f$ be analytic on an open set containing $\sigma(T)$. Suppose for each $\lambda_0$ that $\{ \lambda : f(\lambda) = f(\lambda_0) \}$ is finite. Then $f(\sigma_0(T)) = \sigma_0(f(T))$.

Proof. Suppose $\lambda_0 \in \sigma_0(T)$. Since the spectral mapping theorem implies that $f(\sigma(T)) = \sigma(f(T))$, $f(\lambda_0)$ is either a limit point of $\sigma(f(T))$ or an isolated point. If $f(\lambda_0)$ is a limit point, Corollary 2 implies that $f(\lambda_0) \in \sigma_0(f(T))$. If $f(\lambda_0)$ is isolated, then Theorem 5.71D of Taylor [4] implies that $\sigma = \{ \lambda : f(\lambda) = f(\lambda_0) \} \cap \sigma(T)$ is a finite spectral set of $T$, and the spectral projection associated with $\sigma$ and $T$, $E_\sigma(T)$, equals that associated with $f(\lambda_0)$ and $F_{f(\lambda_0)}(f(T))$, i.e. $E_\sigma(T) = F_{f(\lambda_0)}(f(T))$. Since $\sigma$ is a finite spectral set, this implies that $\lambda_0$ is an isolated point. Corollary 2 implies that $E_{\lambda_0}$ is infinite dimensional. Hence $F_{f(\lambda_0)}$ is infinite dimensional. Thus $f(\lambda_0) \in \sigma_0(f(T))$, or $f(\sigma_0(T)) \subset \sigma_0(f(T))$.

Suppose that $\mu \in \sigma_0(f(T))$. If $\mu$ is a limit point of $\sigma(f(T))$, then since $f(\sigma(T)) = \sigma(f(T))$, there is a limit point $\lambda$ of $\sigma(T)$ such that $f(\lambda) = \mu$. Corollary 2 implies that $\lambda \in \sigma_0(T)$. If $\mu$ is isolated, then, as before, $\sigma = \{ \lambda : f(\lambda) = \mu \} \cap \sigma(T)$ is a nonempty finite spectral set such that $E_\sigma(T) = F_\mu(f(T))$. Since points of $\sigma$ are isolated, $E_\sigma$ is the finite sum of the spectral projections associated with the points of $\sigma$. Since $F_\mu$ is infinite dimensional, one of these projections must be infinite dimensional. Thus there is a $\lambda \in \sigma$ such that $f(\lambda) = \mu$ and $\lambda \in \sigma_0(T)$. Thus, $f(\sigma_0(T)) = \sigma_0(f(T))$. 

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Remark. The above theorems hold if \( P(T) \) and \( Q(T) \) are replaced with finite-dimensional operators that satisfy the defining conditions for these sets.

References


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