TAMING A SURFACE BY PIERCING WITH DISKS

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In [3] and [2], respectively, Bing proves the following two theorems.

**Theorem 1.** A 2-sphere $S$ in $E^3$ is tame from complementary domain $U$ if and only if $U$ is 1-ULC.

**Theorem 2.** If $S$ is a 2-sphere in $E^3$ and $U$ is a complementary domain of $S$ then there exists a 0-dimensional $F$, set $F \subseteq S$ such that $U \cup F$ is 1-ULC. Furthermore if $\{X_i\}$ is a sequence of sets in $S$, each of which is either a tame finite graph or a tame Sierpinski curve, then $F$ may be chosen in $S - \bigcup X_i$.

The above two theorems suggest a procedure for showing that a given condition restricting the embedding of $S$ implies $S$ is tame from $U$. Namely, it may be possible to use the condition to slightly adjust a map $f$ from a disk $D$ into $U \cup F$ so that the new image of $D$ lies entirely in $U$ while $f|_{\partial D}$ is unaltered. The facts that $f(D) \cap F$ is compact 0-dimensional and $F$ lies in $S - \bigcup X_i$ may also be helpful while adjusting $f$.

The above technique is employed in this paper to answer in the affirmative the following question asked in [1] and [5]. Is a 2-sphere in $E^3$ tame if it can be pierced at each arc with a tame disk? Other illustrations of this procedure may be found in [7] and [8].

**Definition.** A disk $D$ is said to pierce sphere $S$ at arc $A$ if $\partial A \subseteq \partial D$, $\operatorname{Int} A \subseteq \operatorname{Int} D$ and the two components of $D - A$ lie in different complementary domains of $S$.

**Definition.** If $J$ is a simple closed curve in 2-sphere $S$ and $U$ is a complementary domain of $S$ then $J$ can be collared from $U$ by an annulus such that $J \cup \partial D$ and $A - J \subseteq U$.

The reader is referred to [2] and [3] for definitions of other terms used in this paper.

**Theorem 3.** If $S$ is a 2-sphere in $E^3$ then $S$ is tame if and only if each simple closed curve in $S$ can be collared from each complementary domain of $S$ by a tame annulus.

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Proof. Let $U$ be a complementary domain of $S$. We will show that $U$ is 1-ULC and apply Theorem 1 to conclude that $S$ is tame from $U$. Suppose $\varepsilon > 0$, then it follows from Theorem 2 that there exists a $\delta > 0$ such that if $f$ is a map of the boundary of a disk $D$ into a $\delta$-subset in $U$ then $f$ may be extended to $D$ so that

1. $f(D)$ is an $\varepsilon/2$ subset of $\overline{U}$,
2. $f(D) \cap S$ is 0-dimensional, and
3. $f(D) \cap S \subseteq \text{Int } A$ where $A$ is an $\varepsilon/2$-annulus in $S$.

It follows from (2) that there exists a homeomorphism $h$ of $\{(x, y) | 1 \leq x^2 + y^2 \leq 4\}$ onto $A$ such that the arc

$$B = h(\{(x, y) | 1 \leq x \leq 2, y = 0\})$$

misses $f(D)$. From the hypothesis there exists a disk $C$ that pierces $S$ at $B$ and also misses $f(D)$. It follows from (1) and (3) that there exists an open set $N$ such that $N \cap S = \text{Int } A$, $N \cap (f(Bd D) \cup Bd C) = \phi$, and $\text{Diam } f(D) \subseteq N < \varepsilon$. The proof will be completed by adjusting the singular disk $f(D)$ in $N$ so that the new image of $D$ lies in $U$.

It follows from the hypothesis that for each $t \in (1, 2)$ there exists a tame annulus $A_t$ such that

4. $A_t \cap S$ is the simple closed curve $J_t = h(\{(x, y) | x^2 + y^2 = t^2\})$, and

5. $A_t \subseteq N \cap \overline{U}$.

We need the following well-known lemma from general topology. An easy proof may be obtained by showing that the space of all continuous functions from an element of $C$ into $X$ is a separable metric space under the sup-norm metric. The elements of $C$ may then be considered as points in this separable metric space.

**Lemma 4.** If $C$ is an uncountable collection of homeomorphic compact subsets of a separable metric space $X$ then there exists a countable subcollection $C'$ of $C$ such that if $Y \in C - C'$ then there exists a sequence $\{Y_i\} \subset C - C'$ which converges homeomorphically to $Y$.

Apply Lemma 4 by letting $C = \{A_t | t \in (1, 2)\}$ and consider an $A_t \in C - C'$. There is a sequence $\{A_{t_i}\} \subset C - C'$ converging homeomorphically to $A_t$; consequently, there exists an integer $i$ such that $A_t$ and $A_{t_i}$ are homeomorphically so close that there exists a singular annulus $B_t$ (resulting from a homotopy) such that

6. $\text{Bd } B_t = (\text{Bd } A_t \cup \text{Bd } A_{t_i}) - S$, and
7. $B_t \subseteq N \cap \overline{U}$.

The union of $A_t$, $B_t$ and $A_{t_i}$ is a singular annulus which has no singularities near its boundary. Dehn's lemma [9] is applied to replace $A_t \cup B_t \cup A_{t_i}$ with a nonsingular annulus $C_t$ with the same boundary.
Hence there exists a countable set $P \subset (1, 2)$ such that if $t \in (1, 2) - P$ there exists a tame annulus $C_t$ such that

(8) $\partial C_t = J_i \cup J_s$ for some $s \in (1, 2)$, and

(9) $\text{Int } C_t \subset N \cap U$.

Let $D_t$ be the annulus lying in $A$ with the same boundary as $C_t$. It is straightforward to show that there exists a countable set $Q \subset (1, 2)$ such that $s \in Q$ whenever

$$J_s \subset \bigcup_{t \in (1, 2) - P} \text{Int } D_t.$$ 

Furthermore there exists a countable set $R \subset (1, 2) - P$ such that

$$\bigcup_{t \in (1, 2) - P} \text{Int } D_t = \bigcup_{t \in R} \text{Int } D_t.$$ 

The set

$$Y = (\bigcup_{s \in Q} J_s) \cup (\bigcup_{t \in R} \partial D_t)$$

is the union of a countable number of tame simple closed curves, so by Theorem 2 there exists a $0$-dimensional set $F \subset S - Y$ such that $U \cup F$ is $1$-ULC.

It follows that there exists an open set $V$ containing $f(D) \cap S$ such that loops in $V \cap U$ can be shrunk to points in $(U \cup F) \cap (N - C)$. It is straightforward to find a finite collection $E_1, \ldots, E_k$ of disjoint disks in $D$ such that $f^{-1}(S) \subset \bigcup_{i=1}^k E_i \subset \text{Int } D$ and $f(\partial D \cup E_i) \subset V \cap U$. The map $f|_{\partial D \cup E_i}$ is extended to a map $f_i: E_i \to (U \cup F) \cap (N - C)$. The maps $f_i$ ($i = 1, \ldots, k$) and $f|_D \cap D \cup E_i$ are pieced together to form a map $g: D \to U \cup F$. Note that $g(D) \cap S \subset \text{Int } A \subset (U \cap \text{Int } D_t) - Y$.

There exists a finite set $T \subset R$ such that $g(D) \cap S \subset \text{Int } \bigcup_{i \in T} D_t$. We assume that $C_t$ and $C_{t'}$ are in general position whenever $t, t' \in R$ and $t \neq t'$. Note that for each $s \in T$ a simple closed curve $K$ in $\text{Int } C_t$, links $\partial D$ if and only if $K$ separates the boundary components of $C_t$ in $C_t'$; thus, a simple closed curve in $\text{Int } C_t \cap \text{Int } C_{t'}$ bounds a disk in $\text{Int } C_t$ if and only if it bounds a disk in $\text{Int } C_{t'}$. Using standard disk and annulus trading techniques we alter the collection $\{C_i\}_{i \in T}$ to form a new finite collection $\{F_i\}_{i=1}^n$ of annuli such that

(10) $\partial F_i = J_i \cup J_s$ for some $s, t \in (1, 2)$,

(11) $\text{Int } F_i \subset N \cap U$,

(12) $\text{Int } F_i \cap \text{Int } F_j = \emptyset$ whenever $i \neq j$, and if $G_i$ is the annulus lying in $A$ with the same boundary as $F_i$ then

(13) $g(D) \cap S \subset \bigcup_{i=1}^n \text{Int } G_i$.

For $i = 1, \ldots, n$ each component of the boundary of $F_i$ links $\partial D$ and $F_i \cap \partial D = \emptyset$; consequently, each simple closed curve in $\text{Int } F_i$
— $C$ bounds a disk in Int $F_i$. Since $g(D) \cap C = \emptyset$, it follows that there exists a finite collection $D_{ij}$ of disjoint disks in Int $F_i$ such that $g(D) \cap F_i \subset \bigcup_j D_{ij}$. The union, $\bigcup_i D_{ij}$, separates $g(Bd D)$ from $g(D) \setminus S$ on $g(D)$ so it follows from the Tietze Extension Theorem as indicated in [4, Lemma 6] that there is a map $f': D \to (U \cap N) \cup (f(D) - N)$ such that $f' | Bd D = g | Bd D = f | Bd D$. Since $\text{Diam } f(D) \cup N < \epsilon$, $\text{Diam } f'(D) < \epsilon$ and we have $U$ is 1-ULC. An application of Theorem 1 completes the proof.

**Theorem 5.** A 2-sphere $S$ in $E^3$ is tame if and only if $S$ can be pierced at each arc with a tame disk.

**Proof.** The lemma after Theorem 4 of [6] shows that the hypothesis of Theorem 3 is satisfied.

R. J. Daverman has recently weakened the hypothesis of Theorem 5 to include the case where $S$ can be pierced at each arc with a singular disk and each arc of $S$ is tame.

**References**


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