APPLICATIONS OF $\epsilon$-ENTROPY TO THE COMPUTATION OF $n$-WIDTHS

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1. The concepts of $\epsilon$-entropy and $n$-width of compact sets in Banach spaces play an important role in approximation theory (see [1], [2], [3] and references therein). The entropy and widths of many compact classes of smooth and analytic functions in various well-known function spaces have been computed.

It is known that entropy and $n$-width are related to each other, e.g., by certain integral inequalities (see [3, p. 164]). There is every reason to believe, however, that in general the behavior of one quantity does not determine in a sharp way the behavior of the other.

The purpose of this paper is to show how the implicit relationship between $n$-width and entropy inherent in certain negative Vituškin type results for nonlinear approximation can be utilized to compute $n$-widths of some classes of smooth functions. As examples we shall compute the $n$-widths for the following classes.

Let $S$ be an $s$-dimensional parallelepiped, and let $\omega$ be a monotone increasing subadditive function which vanishes at zero. We define $\Lambda_{n}^{s}$ in $C(S)$ and $\Lambda_{n}^{p}$ in $L^{p}(S)$, $1 \leq p < \infty$, as

\[ \Lambda_{n}^{s} = \Lambda_{n}^{s}(M_{0}, \cdots, M_{r+1}, S) = \{ f: f \in C(S), \|D^{j}f\|_{\infty} \leq M_{j}, \]
\[ 0 \leq j \leq r, \quad \text{and} \quad \omega(D^{j}f; t) \leq M_{r+1}\omega(t) \} \]

\[ \Lambda_{n}^{p} = \Lambda_{n}^{p}(M_{0}, \cdots, M_{r+1}, S) = \{ f: f \in C^{r}(S), \|D^{j}f\|_{p} \leq M_{j}, \]
\[ 0 \leq j \leq r, \quad \text{and} \quad \omega(D^{j}f; t) \leq M_{r+1}\omega(t) \}, \quad 0 < \alpha \leq 1, \]

where $\omega(D^{j}f; t)$ denotes the modulus of continuity of $D^{j}f$ and $D^{j}f$ denotes an arbitrary partial derivative of $f$ of order $j$.

The $n$-widths of $\Lambda_{n}^{s}$ have been computed elsewhere (cf. [3]) by other means, but the results for $\Lambda_{n}^{p}$ are new.

2. Vituškin-type results. In this section we will state, after some initial definitions, theorems due to Vituškin and Lorentz which say in effect that if the $n$-width of a class is known to be less than $\epsilon$, then $n$ must be at least as large as the $\epsilon$-entropy of that class. These theorems will later be exploited to obtain lower bounds for $n$-widths.
By the \( n \)-width, \( d_n(A) \), of a set \( A \) in a Banach space \( \mathfrak{X} \) we mean the number

\[
d_n(A) = \inf_{\dim \mathcal{M} = n} \sup_{f \in A} \sup_{g \in \mathcal{M}} \|f - g\|
\]

where \( \mathcal{M} \subseteq \mathfrak{X} \), and by the \( \varepsilon \)-entropy \( H_\varepsilon(B) \) in \( \mathfrak{X} \) we mean the logarithm of the minimum number of sets of diameter \( \leq 2\varepsilon \) whose union contains \( B \). Finally, \( \lambda(\varepsilon) \approx \mu(\varepsilon) \) (weak asymptotic equivalence) is to mean that \( \lambda = O(\mu) \) and \( \mu = O(\lambda) \) as \( \varepsilon \to 0 \).

We state first the theorem of Vituškin [1, Theorem 12, p. 928] which, though in its original form deals with nonlinear approximation, is here specialized to the case of linear approximation of the class \( \Lambda_\varepsilon^{*\omega} \).

**Theorem 2.1 (Vituškin).** Consider the class \( \Lambda_\varepsilon^{*\omega} \subseteq C(S) \). If \( d_n(\Lambda_\varepsilon^{*\omega}) \leq \varepsilon \) then

\[
n \geq c_1 H_\varepsilon(\Lambda_\varepsilon^{*\omega}).
\]

Lorentz [1, Theorem 6, p. 915] has proved analogous results for arbitrary separable Banach spaces:

**Theorem 2.2 (Lorentz).** Let \( A \) be an arbitrary compact set in a separable Banach space \( \mathfrak{X} \) and let \( d_n(A) < \varepsilon \). If \( \forall q \), \( 0 < q < 1 \), \( \exists \alpha > 0 \) \( \exists H_\varepsilon^{q\omega}(A) \leq q H_\varepsilon(A) \), then

\[
n \geq c_1 H_\varepsilon^{q\omega}(A) - c_2.
\]

Notice that for sufficiently small \( \varepsilon \) (i.e. sufficiently large \( n \)) \( 2.2 \) may be rewritten as

\[
n \geq c_3 H_\varepsilon^{q\omega}(A).
\]

3. \( n \)-widths of \( \Lambda_\varepsilon^{*\omega} \). In this section we illustrate our method by obtaining the following

**Theorem 3.1 (cf. [3, p. 135]).** The \( n \)-width of the class \( \Lambda_\varepsilon^{*\omega} \subseteq C(S) \) is given by

\[
d_n(\Lambda_\varepsilon^{*\omega}) \approx n^{-r/s} \omega(n^{-1/s}).
\]

**Proof.** By the classical Jackson theorem for several variables [3, Theorem 8, p. 90] if \( n^{1/s} \) is an integer we can find a polynomial \( P_n \) of degree \( n^{1/s} - 1 \) in each of its \( s \) variables such that

\[
\|f - P_n\|_\omega \leq M_s n^{-r/s} \omega(n^{-1/s})
\]

for every \( f \in \Lambda_\varepsilon^{*\omega} \). It follows from this using the subadditivity and monotonicity properties of \( \omega \) that
\[ d_n(\Lambda^*_{\omega}) = O\left(n^{-\tau/\delta}\omega(n^{-1/\delta})\right) \quad \text{as } n \to \infty. \]

To obtain a lower bound for \( d_n \) we notice that by Theorem 2.1 if \( n < c_1 H_\delta(\Lambda^*_{\omega}) \) then \( d_n(\Lambda^*_{\omega}) \geq \varepsilon \). Since (cf. [1, p. 920])

\[ \frac{c_2}{\delta(\beta_0)^s} \leq H_\delta(\Lambda^*_{\omega}) \leq \frac{c_3}{\delta(\gamma_0)^s} \]

where \( \delta = \delta(\eta) \) is defined by the equation \( \delta \omega(\delta) = \eta \) and \( c_2, c_3, \beta, \gamma \)
are positive constants, then any solution \( \varepsilon_n \) of

\[ n = \frac{c_1 c_2}{2\delta(\beta_0)^s} \]

provides a lower bound for \( d_n(\Lambda^*_{\omega}) \).

This equation may be rewritten as

\[ \frac{\delta(\beta_0)^s}{c_4} = \left(\frac{n}{c_4}\right)^{-1/s} \quad (c_4 = c_1 c_2/2), \]

and hence recalling that \( \delta(\beta_0)\omega(\delta(\beta_0)) = \beta_0 \) we obtain

\[ \beta_0 = \left(\frac{n}{c_4}\right)^{-1/s} \omega\left(\left(\frac{n}{c_4}\right)^{-1/\delta}\right). \]

But then

\[ d_n(\Lambda^*_{\omega}) \geq \varepsilon_n \geq c_6 n^{-\tau/s}\omega(n^{-1/\delta}) \]

where

\[ c_6 = \frac{c_4^{1/\delta}}{\beta([c_4^{1/\delta}] + 1)}. \]

This completes the proof of Theorem 3.1.

4. \( n \)-widths of \( \Lambda^p_{\alpha^p} \).

**Theorem 4.1.** The \( n \)-width of the class \( \Lambda^p_{\alpha^p} \subset L^p(S) \) is given by

\[ d_n(\Lambda_{\alpha^p}) \approx n^{-(r+\alpha)/s} \quad (1 \leq p < \infty). \]

**Proof.** Since \( \|g\|_p \leq K\|g\|_\alpha \) for all \( g \in L^p(S) \) we have, by the \( s \)-dimensional Jackson theorem,

\[ d_n(\Lambda_{\alpha^p}) \leq c n^{-(r+\alpha)/s} \]

for the \( L^p \) \( n \)-width of \( \Lambda_{\alpha^p} \). To compute a lower bound we apply
Theorem 2.2 with $A = \Lambda_{\sigma_0}$ and $\mathcal{X} = L^p$. The entropy of the class $\Lambda_{\sigma_0}^p$ is given by [1, p. 921].

$$H_{\epsilon}(\Lambda_{\sigma_0}^p) = e^{-\epsilon/(r+\alpha)}.$$ 

Since $H_{\epsilon}$ satisfies the hypotheses of Theorem 2.2 the result follows as in §3.

References