APPLICATIONS OF $\varepsilon$-ENTROPY TO THE COMPUTATION OF $n$-WIDTHS

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1. The concepts of $\varepsilon$-entropy and $n$-width of compact sets in Banach spaces play an important role in approximation theory (see [1], [2], [3] and references therein). The entropy and widths of many compact classes of smooth and analytic functions in various well-known function spaces have been computed.

It is known that entropy and $n$-width are related to each other, e.g., by certain integral inequalities (see [3, p. 164]). There is every reason to believe, however, that in general the behavior of one quantity does not determine in a sharp way the behavior of the other.

The purpose of this paper is to show how the implicit relationship between $n$-width and entropy inherent in certain negative Vituškin type results for nonlinear approximation can be utilized to compute $n$-widths of some classes of smooth functions. As examples we shall compute the $n$-widths for the following classes. Let $S$ be an $s$-dimensional parallelepiped, and let $\omega$ be a monotone increasing subadditive function which vanishes at zero. We define $\Lambda_{\omega}^{*}$ in $C(S)$ and $\Lambda_{\omega}^{p}$ in $L^{p}(S), 1 \leq p < \infty$, as

$$\Lambda_{\omega}^{*} = \Lambda_{\omega}^{*}(M_{0}, \cdots, M_{r+1}, S) = \{ f : f \in C(S), \| D^{j}f \|_{\infty} \leq M_{j}, 0 \leq j \leq r, \text{ and } \omega(D^{j}f; t) \leq M_{r+1}\omega(t) \}$$

$$\Lambda_{\omega}^{p} = \Lambda_{\omega}^{p}(M_{0}, \cdots, M_{r+1}, S) = \{ f : f \in C^{r}(S), \| D^{j}f \|_{p} \leq M_{j}, 0 \leq j \leq r, \text{ and } \omega(D^{j}f; t) \leq M_{r+1}\omega^{a}(t), 0 < \alpha \leq 1 \},$$

where $\omega(D^{j}f; t)$ denotes the modulus of continuity of $D^{j}f$ and $D^{j}f$ denotes an arbitrary partial derivative of $f$ of order $j$.

The $n$-widths of $\Lambda_{\omega}^{*}$ have been computed elsewhere (cf. [3]) by other means, but the results for $\Lambda_{\omega}^{p}$ are new.

2. Vituškin-type results. In this section we will state, after some initial definitions, theorems due to Vituškin and Lorentz which say in effect that if the $n$-width of a class is known to be less than $\varepsilon$, then $n$ must be at least as large as the $\varepsilon$-entropy of that class. These theorems will later be exploited to obtain lower bounds for $n$-widths.

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By the n-width, $d_n(A)$, of a set $A$ in a Banach space $\mathcal{X}$ we mean the number

$$d_n(A) = \inf_{\dim \mathcal{M} = n} \sup_{f \in A} \inf_{g \in \mathcal{M}} \|f - g\|$$

where $\mathcal{M} \subset \mathcal{X}$, and by the $\epsilon$-entropy $H_\epsilon(B)$ in $\mathcal{X}$ we mean the logarithm of the minimum number of sets of diameter $\leq 2\epsilon$ whose union contains $B$. Finally, $\lambda(\epsilon) \approx \mu(\epsilon)$ (weak asymptotic equivalence) is to mean that $\lambda = O(\mu)$ and $\mu = O(\lambda)$ as $\epsilon \to 0$.

We state first the theorem of Vitushkin [1, Theorem 12, p. 928] which, though in its original form deals with nonlinear approximation, is here specialized to the case of linear approximation of the class $\Lambda_{nw}$.

**Theorem 2.1 (VITUSHKIN).** Consider the class $\Lambda_{nw} \subset C(S)$. If $d_n(\Lambda_{nw}) < \epsilon$ then

$$n \geq c_1 H_\epsilon(\Lambda_{nw}).$$

Lorentz [1, Theorem 6, p. 915] has proved analogous results for arbitrary separable Banach spaces:

**Theorem 2.2 (LORENTZ).** Let $A$ be an arbitrary compact set in a separable Banach space $\mathcal{X}$ and let $d_n(A) < \epsilon$. If $\forall q, 0 < q < 1, \exists \epsilon_0 > 0 \ni H_{\epsilon_0}(A) \leq q H_\epsilon(A)$, then

$$n \geq c_1 H_{\epsilon_0}(A) - c_2.$$  

Notice that for sufficiently small $\epsilon$ (i.e. sufficiently large $n$) (2.2) may be rewritten as

$$n \geq c_3 H_{\epsilon_0}(A).$$

3. $n$-widths of $\Lambda_{nw}$. In this section we illustrate our method by obtaining the following

**Theorem 3.1 (cf. [3, p. 135]).** The $n$-width of the class $\Lambda_{nw} \subset C(S)$ is given by

$$d_n(\Lambda_{nw}) \approx n^{-r/s} \omega(n^{-1/s}).$$

**Proof.** By the classical Jackson theorem for several variables [3, Theorem 8, p. 90] if $n^{1/s}$ is an integer we can find a polynomial $P_n$ of degree $n^{1/s} - 1$ in each of its $s$ variables such that

$$\|f - P_n\|_\omega \leq M_s n^{-r/s} \omega(n^{-1/s})$$

for every $f \in \Lambda_{nw}$. It follows from this using the subadditivity and monotonicity properties of $\omega$ that

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\[ d_n(\Lambda^*_{\omega}) = O(n^{-r/s} \omega(n^{-1/s})) \quad \text{as } n \to \infty. \]

To obtain a lower bound for \( d_n \) we notice that by Theorem 2.1 if \( n < c_1 H(\Lambda^*_{\omega}) \) then \( d_n(\Lambda^*_{\omega}) \geq \varepsilon \). Since (cf. [1, p. 920])

\[ \frac{c_2}{\delta(\beta_\varepsilon)^s} \leq H_*(\Lambda^*_{\omega}) \leq \frac{c_3}{\delta(\gamma_\varepsilon)^s}, \]

where \( \delta = \delta(\eta) \) is defined by the equation \( \delta \omega(\delta) = \eta \) and \( c_2, c_3, \beta, \gamma \) are positive constants, then any solution \( \varepsilon_n \) of

\[ n = \frac{c_1 c_2}{2 \delta(\beta_\varepsilon)^s} \]

provides a lower bound for \( d_n(\Lambda^*_{\omega}) \).

This equation may be rewritten as

\[ \delta(\beta_\varepsilon_n) = \left( \frac{n}{c_4} \right)^{-1/s} \quad (c_4 = c_1 c_2/2), \]

and hence recalling that \( \delta^*(\beta_\varepsilon_n) \omega(\delta(\beta_\varepsilon_n)) = \beta_\varepsilon_n \), we obtain

\[ \beta_\varepsilon_n = \left( \frac{n}{c_4} \right)^{-r/s} \omega \left( \left( \frac{n}{c_4} \right)^{-1/s} \right). \]

But then

\[ d_n(\Lambda^*_{\omega}) \geq \varepsilon_n \geq c_5 n^{-r/s} \omega(n^{-1/s}) \]

where

\[ c_5 = \frac{c_4^{r/s}}{\beta([c_4^{-1/s}] + 1)}. \]

This completes the proof of Theorem 3.1.

4. \( n \)-widths of \( \Lambda_{\omega}^{sp} \).

**Theorem 4.1.** The \( n \)-width of the class \( \Lambda_{\omega}^{sp} \subseteq L^p(S) \) is given by

\[ d_n(\Lambda_{\omega}^{sp}) \approx n^{-(r+\alpha)/s} \quad (1 \leq p < \infty). \]

**Proof.** Since \( \|g\|_p \leq K\|g\|_{\omega} \) for all \( g \in L^p(S) \) we have, by the \( s \)-dimensional Jackson theorem,

\[ d_n(\Lambda_{\omega}^{sp}) \leq cn^{-(r+\alpha)/s} \]

for the \( L^p \) \( n \)-width of \( \Lambda_{\omega}^{sp} \). To compute a lower bound we apply
Theorem 2.2 with $A = A_{<0}$ and $X = L^p$. The entropy of the class $A_{<0}^{sp}$ is given by [1, p. 921].

$$H_s(A_{<0}^{sp}) = e^{-s/(r+\alpha)}.$$ 

Since $H_s$ satisfies the hypotheses of Theorem 2.2 the result follows as in §3.

**References**