ON A WEAKLY CONVERGENT SEQUENCE OF NORMAL FUNCTIONALS ON A VON NEUMANN ALGEBRA

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G. F. Dell'Antonio [1] recently has discussed weakly convergent sequences of normal states of von Neumann algebras and proved that every weakly convergent sequence of normal states of a factor of type I converges also uniformly. Moreover, he has shown that this statement is not true for a factor of type II. The purpose of this note is to investigate when a weakly convergent sequence of normal states converges also uniformly in the case of type II factors. We shall confine ourselves to the class of normal generalized irreducible functionals on a factor of type II. Then the generalized irreducibility of functionals makes it possible to find a simple and relevant condition for our problem.

Throughout this paper, for convenience functional will always mean a positive linear functional on a von Neumann algebra. Let us recall that a functional \( \rho \) on a von Neumann algebra \( M \) is said to be generalized irreducible on \( M \) if whenever \( \omega \) is a functional on \( M \) such that that \( \omega \leq \lambda \rho \) for some positive constant \( \lambda \) (i.e., \( \omega(A) \leq \lambda \rho(A) \) for all positive operators \( A \) in \( M \)), there exists a positive operator \( B \) in \( M \) such that \( \omega(A) = \rho(AB) \) for all \( A \in M \). As is well known, every normal trace of a finite von Neumann algebra is generalized irreducible (see [4, Lemma 14.1]). We say that a sequence \( \{\rho_n\} \) of functionals on a von Neumann algebra \( M \) is bounded from below by a functional \( \rho \) on \( M \) if \( \rho \leq \rho_n \) for all \( n \). Then we shall prove the following.

**Theorem.** Let \( \{\rho_n\} \) be a sequence of normal generalized irreducible functionals on a semifinite factor \( M \) bounded from below by a nonzero normal functional \( \omega \) on \( M \). If \( \rho_n \) converges weakly to a normal generalized irreducible functional \( \rho \) on \( M \), then \( \rho_n \) converges also uniformly to \( \rho \).


1. In what follows, \( M \) will denote a semifinite factor on a Hilbert space \( H \) with inner product \( \langle \cdot, \cdot \rangle \). First we are concerned with a representation theorem of a normal generalized irreducible functional on \( M \) which is essentially due to Halpern [3, Proposition 3.1].

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Lemma 1. Let \( \rho \) be a normal generalized irreducible functional on \( M \). If \( \rho \) is faithful on \( M \), then \( \rho \) is a trace of \( M \).

Proof. Let us consider the faithful representation \( \pi \) of \( M \) defined by \( \rho \). Then the representation space is the completion \( K \) of the pre-Hilbert space \( M \) with inner product \( \langle A, B \rangle = \rho(B^*A) \), and \( \rho \) has the form \( \rho(A) = \langle \pi(A)\xi, \xi \rangle \), where \( \xi \) is a cyclic vector of \( \pi(M) \), i.e., \( [\pi(M)\xi] = K \). Moreover, \( [\pi(M)\xi] = K \) since \( \rho \) is faithful. Thus we may assume without loss of generality that there exists a vector \( \phi \in H \) such that

\[
\rho(A) = \langle A\phi, \phi \rangle \quad \text{and} \quad H = [M\phi] = [M'\phi].
\]

Then it follows from [3, Proposition 3.1] that \( \phi \) is a trace vector of \( M \), that is, \( \rho \) is a trace of \( M \).

Lemma 2. Let \( \rho \) be a normal functional on \( M \) and let \( \tau \) be a faithful normal trace defined on a two-sided ideal \( \mathfrak{M} \) of \( M \) containing all of the finite projections in \( M \). Then \( \rho \) is generalized irreducible if and only if it is represented in the form

\[
\rho(A) = \tau(\lambda EA) \quad \text{for all } A \in M,
\]

where \( E \) is the support of \( \rho \) and \( \lambda \) is a positive constant.

Proof. The "if" part follows immediately from [4, Lemma 14.1].

For each \( A \in M \), we denote by \( A_E \) the restriction of \( EAE \) to \( EH \) and by \( M_E \) the restriction of \( EME \) to \( EH \). Then \( \rho \) induces a faithful normal generalized irreducible functional \( \bar{\rho} \) on \( M_E \) by restriction. Indeed, if \( \omega \) is a functional on the factor \( M_E \) such that \( \omega \leq \mu \bar{\rho} \) for some positive constant \( \mu \), then the functional \( \omega \) on \( M \) defined by \( \omega(A) = \bar{\omega}(A_E) \in M \) is bounded by \( \mu \rho \). Thus there is a positive operator \( B \in M \) such that \( \omega(A) = \rho(AB) \) for all \( A \in M \), and so

\[
\bar{\omega}(A_E) = \omega(EAE) = \rho(EAEB) = \rho(EBE) + \rho(EB(I - E)) = \rho(EBE) = \rho(A_E B_E).
\]

Now it turns out from Lemma 1 that \( \bar{\rho} \) is a trace of \( M_E \). Since \( M_E \) is a finite factor, there is a positive constant \( \lambda \) such that \( \bar{\rho} = \lambda \tau \), where \( \tau \) is the faithful normal trace of \( M_E \) induced by the restriction of \( \tau \) on \( EME \). This means that \( \rho(EAE) = \lambda \tau(EAE) \) for all \( A \in M \). Therefore we have

\[
\rho(A) = \rho(EAE) = \tau(\lambda EA) \quad \text{for all } A \in M.
\]

Remark. As we have seen above, the support \( E \) of a normal generalized irreducible functional on \( M \) is necessarily a finite projection in \( M \), i.e., \( E \in \mathfrak{M} \).
The following lemma is elementary, but is of fundamental importance.

**Lemma 3.** Let \( \{\rho_n\} \) be a sequence of functionals on a von Neumann algebra. If \( \{\rho_n\} \) converges weakly to 0, then \( \rho_n \) converges uniformly to 0.

In fact, \( \|\rho_n\| = \rho_n(I) \to 0 \) as \( n \to \infty \).

2. **Proof of Theorem.** By Lemma 2, \( \rho_n \) and \( \rho \) are expressed in the forms \( \rho_n(A) = \lambda_n \tau(E_nA) \) and \( \rho(A) = \lambda \tau(EA) \), where \( E_n \) and \( E \) are the supports of \( \rho_n \) and \( \rho \) respectively. Let \( F \) be the support of \( \omega \). Then \( F \) is nonzero and \( F \subseteq E_n, E \) by the hypothesis. Since \( \rho_n(F) \to \rho(F) \), \( \lambda_n \tau(F) \to \lambda \tau(F) \) and hence \( \lambda_n \to \lambda \) as \( n \to \infty \).

Now let us consider the factor \( \tilde{M} = M_E \) obtained by restricting \( EM_E \) to \( EH \) and \( \tilde{A} \) denote an operator in \( \tilde{M} \) which is the restriction of \( EA_E \) to \( EH \). \( \tilde{\rho}_n \) and \( \tilde{\tau} \) denote the functionals on \( \tilde{M} \) induced by the restrictions of \( \rho_n \), \( \rho \) and \( \tau \) on \( EM_E \) respectively. That is, \( \tilde{\rho}_n(\tilde{A}) = \rho_n(EM_E) \) and \( \tilde{\rho}(\tilde{A}) = \rho(EM_E) \). Then it is easily seen that \( \tilde{\rho}_n(\tilde{A}) = \lambda_n \tilde{\tau}(\tilde{E}_n \tilde{A}) \) and \( \tilde{\rho}(\tilde{A}) = \lambda \tilde{\tau}(\tilde{E} \tilde{A}) \). Here we should notice that \( \tilde{E}_n \) are positive operators in \( \tilde{M} \) such that \( \tilde{E}_n \leq \tilde{E} \). We shall show that \( \tilde{\rho}_n \to \tilde{\rho} \) uniformly as \( n \to \infty \). To prove this it is enough to consider the case when \( \tilde{M} \) is standard. In this case, as is well known, \( \tilde{\tau} \) is expressible as the form \( \tilde{\tau}(\tilde{A}) = \langle \tilde{A} \phi, \phi \rangle \), where \( \phi \) is a trace vector of \( \tilde{M} \) such that \( [\tilde{M} \phi] = [\tilde{M} \phi] = EH \). Then it follows from the hypothesis that \( \langle \lambda_n \tilde{E}_n \tilde{A} \phi, \tilde{A} \phi \rangle \to \langle \lambda \tilde{E} \tilde{A} \phi, \tilde{A} \phi \rangle \) for each \( \tilde{A} \in \tilde{M} \). Since \( \{\lambda_n \tilde{E}_n\} \) is bounded and \( EH = [\tilde{M} \phi] \), the sequence \( \{\lambda_n \tilde{E}_n\} \) of positive operators converges weakly to \( \lambda \tilde{E} \) in \( \tilde{M} \). Thus, having recalled that \( \lambda_n \to \lambda \), \( \tilde{E}_n \to \tilde{E} \) weakly. This means that the sequence \( \{ \tilde{E} - \tilde{E}_n \} \) of positive operators converges weakly to 0, and so it converges also strongly to 0. That is to say, \( \tilde{E}_n \to \tilde{E} \) strongly. Hence we can conclude that \( \lambda_n \tilde{E}_n \to \lambda \tilde{E} \) strongly. Consequently, for each \( \epsilon > 0 \), there is a positive integer \( N \) such that \( \|\lambda_n \tilde{E}_n - \lambda \tilde{E} \phi\| < \epsilon/\|\phi\| \) for all \( n \geq N \). Then

\[
\|\tilde{\rho}_n - \tilde{\rho}\| = \sup_{\|\tilde{A}\| = 1} |\tilde{\rho}_n(\tilde{A}) - \tilde{\rho}(\tilde{A})| \\
= \sup_{\|\tilde{A}\| = 1} |\langle \lambda_n \tilde{E}_n - \lambda \tilde{E} \phi, \tilde{A} \phi \rangle| \leq \|\lambda_n \tilde{E}_n - \lambda \tilde{E} \phi\| \|\phi\| < \epsilon
\]

for all \( n \geq N \).

Let \( E' = I - E \) and let \( \tilde{M} \) be the factor on \( E'H \) obtained by the restriction of \( E'M'E' \) on \( E'H \). We denote by \( \tilde{A} \) the restriction of \( E'A'E' \) to \( E'H \), and by \( \tilde{\rho}_n \) and \( \tilde{\rho} \) the functionals on \( \tilde{M} \) induced by restricting \( \rho_n \) and \( \rho \) on \( E'M'E' \) respectively. Then, since \( E \) is the support of \( \rho, \tilde{\rho} = 0 \). Thus we have
\[\begin{align*}
|\rho_n(A) - \rho(A)| & \leq |\rho_n(EAE) - \rho(EAE)| + |\rho_n(E'AE)| \\
& \quad + |\rho_n(EAE')| + |\rho_n(E'AE')| \\
& \leq \|\bar{p}_n - \bar{p}\| \|A\| + |\rho_n(EAE)| + |\rho_n(EAE')| \\
& \quad + \|\bar{p}_n\| \|A\|.
\end{align*}\]

Here
\[|\rho_n(EAE)| \leq \rho_n(E)^{1/2}\rho_n(EA^*AE)^{1/2} \leq \rho_n(E)^{1/2}\|\bar{p}_n\|^{1/2}\|(A^*A)^{-1}\|^{1/2}\]
and
\[|\rho_n(EAE')| \leq \rho_n(E)^{1/2}\rho_n(E'AE')^{1/2} \leq \rho_n(E)^{1/2}\|\bar{p}_n\|^{1/2}\|(A^*A)^{-1}\|^{1/2}.
\]

Since \(\rho_n \rightarrow \rho\) weakly, there is a constant \(K\) such that \(\rho_n(E)^{1/2} \leq K\) for all \(n\). Thus from (1) we have the following
\[\|\rho_n - \rho\| = \sup_{\|A\|=1} |\rho_n(A) - \rho(A)| \leq \|\bar{p}_n - \bar{p}\| + \rho_n(E)^{1/2}\|\bar{p}_n\|^{1/2} + K\|\bar{p}_n\|^{1/2} + \|\bar{p}_n\|.
\]
By what we have proved, \(\|\bar{p}_n - \bar{p}\| \rightarrow 0\), and so \(\{\|\bar{p}_n\|\}\) is bounded. Further, \(\rho_n(E)^{1/2} \rightarrow \rho(E)^{1/2} = 0\) and by Lemma 3, \(\|\rho_n\| \rightarrow 0\). Thus it follows from (2) that \(\|\rho_n - \rho\| \rightarrow 0\) as \(n \rightarrow \infty\).

3. Finally we shall show that a weakly convergent sequence of normal generalized irreducible functionals does not necessarily converge uniformly. An example is obtained by a slight modification of the example given in [1, p. 422]. Let \(M\) be a factor of type II_1 and let \(\tau\) be a (normalized) faithful normal trace of \(M\). Denote by \(K\) the completion of the pre-Hilbert space \(M\) with inner product \((A, B) = \tau(B^*A)^{1/2}\). Then, by [1, Lemma 5], there is an orthonormal sequence (considered as vectors in \(K\)) in \(M\) consisting of selfadjoint unitary operators \(U_n\) \((n = 1, 2, \ldots)\) such that \(\tau(U_n) = 0\). That is, \(U_n^2 = I\) and \(\tau(U_n U_m) = \delta_{nm}\). Define a sequence of projections by \(E_n = \frac{1}{2}(I - U_n)\), and put \(\rho_n(A) = \tau(E_n A) (A \in M)\). Then all \(\rho_n\) are generalized irreducible on \(M\) by Lemma 2. Since \(\{U_n\}\) is orthonormal in \(K\),
\[\sum_n |\langle A, U_n \rangle|^2 \leq \|[A]\| < \infty,
\]
where \(\|[A]\| = \langle A, A \rangle^{1/2}\). Thus \(\langle A, U_n \rangle \rightarrow 0\) as \(n \rightarrow \infty\), in other words, \(\tau(U_n A) \rightarrow 0\). Namely, \(\rho_n(A) = \frac{1}{2}\tau(A - U_n A) \rightarrow \frac{1}{2}\tau(A)\) for each \(A \in M\). This means that \(\rho_n\) converges weakly to a normal generalized irreducible functional \(\frac{1}{2}\tau\). But, having noticed that \(\rho_n(U_n) = -\frac{1}{2}\) and \(\frac{1}{2}\tau(U_n) = 0\),
\[\|\rho_n - \frac{1}{2}\tau\| \geq \frac{1}{2}\quad \text{for all } n.
\]
Thus $\rho_n$ does not converge uniformly to $\frac{1}{2}\tau$. Indeed, $\{\rho_n\}$ is not bounded from below by a nonzero normal functional on $M$. This fact may immediately be verified as follows: $\tau(E_n) = \frac{1}{2}$ for all $n$ and hence

$$\tau(E_1 \cap E_2 \cap \cdots \cap E_n)^2 \leq \left(\frac{1}{2}\right)^n$$

is proved by induction. Thus there does not exist a nonzero projection $F$ in $M$ such that $F \leq E_n$ for all $n$.

References


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