A NOTE ON THE INDEX OF A G-MANIFOLD\textsuperscript{1}

HSU-TUNG KU AND MEI-CHIN KU\textsuperscript{2}

1. Introduction. By a $G$-manifold $M$ we mean a compact Lie group $G$ acting differentiably and preserving orientation on an oriented smooth manifold $M$. The purpose of this paper is to study the index of a $4k$-dimensional $G$-manifold.

Let $M^{4k}$ be a $4k$-dimensional $G$-manifold with or without boundary. The cup-product defines a nondegenerate quadratic form $f$ on $H^{2k}(M, \partial M; R)$, where $R$ is the field of real numbers. Let $H^{+}$ (resp. $H^{-}$) be the maximal subspace on which this form is positive (resp. negative) definite. The subspaces $H^{+}$ and $H^{-}$ are $G$-modules over $R$, hence $H^{+} - H^{-} \subseteq RO(G)$. The index of $M$ is defined to be [4]

\[ \tau(M) = \dim H^{+} - \dim H^{-} \]

Now, for any element $g \in G$, the Atiyah-Singer signature $\tau(g, M)$ is defined by evaluating the character of $H^{+} - H^{-}$ on $g$. Hence

\[ \tau(g, M) = \text{Trace} (g^{*} | H^{+}) - \text{Trace} (g^{*} | H^{-}) \]

where $g^{*}: H^{2k}(M, \partial M; R) \rightarrow H^{2k}(M, \partial M; R)$.

For any $G$-module $V$ over $R$, let

\[ V^{g} = \{ v \in V | gv = v \text{ for all } g \in G \} \]

We define the $G$-index $\tau^{g}(M)$ as follows:

\[ \tau^{g}(M) = \dim(H^{+})^{g} - \dim(H^{-})^{g} \]

From the definition, we have

Proposition 1.1. Suppose $M$ is a $4k$-dimensional $G$-manifold. If $G$ acts trivially on $H^{2k}(M, \partial M; R)$, then $\tau^{g}(M) = \tau(M)$.

2. Main theorems. The relationship between the index $\tau^{g}(M)$ and Atiyah-Singer signature $\tau(g, M)$ is the following:

Theorem 2.1. Let $M$ be a $4k$-dimensional $G$-manifold. Then $\tau^{g}(M) = \int_{g} \tau(g, M) dg$.

Proof. First, we show that

\[ \dim(H^{\pm})^{g} = \int_{g} \text{Trace} (g^{*} | H^{\pm}) dg. \]

Received by the editors October 4, 1968 and, in revised form, January 13, 1969.

\textsuperscript{1} This paper is based on [5].

\textsuperscript{2} The authors are indebted to the referee for some suggestions.

600
To prove this, let
\[ \phi(x) = \int_G g^*(x) \, dg \quad \text{for } x \in H^\pm. \]

Then \( \phi \cdot g^* = g^* \cdot \phi = \phi \) for any \( g \in G \), and \( \phi^* = \phi \), and so \( \text{Im } \phi = (H^\pm)^G \).
Thus \( x \in (H^\pm)^G \) if and only if \( \phi(x) = x \). Hence
\[ \dim((H^\pm)^G) = \text{Trace}(\phi) = \int_G \text{Trace}(g^* | H^\pm) \, dg. \]

Combine this result together with the definition of \( \tau^G(M) \), we have
\[ \tau^G(M) = \int_G \{ \text{Trace}(g^* | H^+) - \text{Trace}(g^* | H^-) \} \, dg \]
\[ = \int_G \tau(g, M) \, dg. \]

**Theorem 2.2.** Let \( G \) be a finite group of order \( q \) acting differentiably and preserving orientation on an oriented smooth \( 4k \)-dimensional closed manifold \( M \) so that the orbit space \( M/G \) has a fundamental class and \( \tau(M/G) \) is defined. Then \( \tau^G(M) = \tau(M/G) \).

**Proof.** We consider the quadratic form \( \tilde{f} \) on \( H^{2k}(M/G; \mathbb{R}) \) defined by cup-product. By [2, p. 38] we have
\[ H^*(M/G; \mathbb{R}) \cong H^*(M; \mathbb{R})^G, \]
where \( \pi : M \to M/G \) is the orbit map. Let \( f^G \) be the quadratic form on \( H^*(M; R)^G \) defined by cup-product. Then (cf. [3, p. 37])
\[ f^G(\pi^* x, \pi^* y) = (\pi^* x \cup \pi^* y)[M] = (x \cup y)_{\pi_*}[M] \]
\[ = q(x \cup y)[M/G] = q\tilde{f}(x, y), \]
where \([M]\) and \([M/G]\) denote the fundamental classes of \( M \) and \( M/G \) respectively. Hence the quadratic forms \( f \) and \( f^G \) have the same index, whence \( \tau^G(M) = \tau(M/G) \).

**Theorem 2.3.** Let \( G \) be a finite group of order \( q \) and \( M \) be a \( 4k \)-dimensional closed \( G \)-manifold. If \( G \) acts freely on \( M \), then
\[ \tau(M) = q\tau(M/G). \]

**Proof.** We note that the tangent bundle to \( M \) is induced by \( \pi \) from the tangent bundle to \( M/G \). Thus \( \pi^* \) maps the Pontrjagin classes \( p_i(M/G) \) of \( M/G \) onto the Pontrjagin classes \( p_i(M) \) of \( M \). Moreover
\[ \pi_*[M] = q[M/G]. \] Hence the Pontrjagin number of \( M \) is \( q \) times the corresponding Pontrjagin number of \( M/G \). By Hirzebruch index theorem \([4]\), we have \( \tau(M) = q \tau(M/G) \). Hence the result follows.

**Corollary 2.4.** Let \( G \) be a finite group of order \( q \) acting freely on a 4k-dimensional closed \( G \)-manifold \( M \). Then \( \tau(M) = q \tau^G(M) \).

**Proof.** By Theorem 2.2 and Theorem 2.3.

**Corollary 2.5.** Let \( G \) be a nontrivial finite group acting freely on a 4k-dimensional closed \( G \)-manifold \( M \) such that \( G \) acts trivially on \( H^{2k}(M; \mathbb{R}) \). Then \( \tau(M) = 0 \). In particular, if \( G \) is a compact connected Lie group acting freely on \( M^{4k} \), then \( \tau(M) = 0 \).

**Proof.** By Proposition 1.1 and Corollary 2.4.

**References**