A NOTE ON SEMIFREE ACTIONS OF $S^1$
ON HOMOTOPY SPHERES

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1. Introduction. Let $(S^1, \Sigma^{n+2k}, \Sigma^n)$, $k \geq 2$, denote a semifree differentiable action of $S^1$ on homotopy $(n+2k)$-sphere $\Sigma^{n+2k}$ with the fixed point set the homotopy $n$-sphere $\Sigma^n$, that is, $S^1$ acts freely outside $\Sigma^n$. We shall call $\Sigma^n$ untwisted if the normal bundle of $\Sigma^n$ in $\Sigma^{n+2k}$ is trivial [2]. The purpose of this note is to study the semifree differentiable actions of $S^1$ on homotopy spheres with homotopy spheres as untwisted fixed point sets. In fact, we shall establish the following theorems by using the results of R. Lee [7].

**Theorem 1.** Let $n \equiv 3 \pmod{4}$ and $4k - 1 \leq n$. If the homotopy sphere $\Sigma^{n+2k}$ admits a semifree differentiable $S^1$ action with untwisted fixed point set $\Sigma^n$, then $\Sigma^{n+2k}$ admits infinitely many differentiably distinct, semifree, differentiable $S^1$ actions with untwisted fixed point set $\Sigma^n$.

**Theorem 2.** There are an infinite number of distinct semifree $S^1$ actions on $S^{17}$ with every element of $32\theta_{11}$ as untwisted fixed point set. For notation $\theta_n$, see [4, p. 504].

2. Proofs of the theorems.

**Definition 1.** The standard (semifree) action of $S^1$ on $S^{n+2k}$ with untwisted fixed point set $S^n$ is defined as follows: Write

$$S^{n+2k} = \left\{ (x_1, \ldots, x_{n+1}, z_1, \ldots, z_k) \in \mathbb{R}^{n+1} \times \mathbb{C}^k \mid \sum_{i=1}^{n+1} x_i^2 + \sum_{j=1}^k |z_j|^2 = 1 \right\}$$

For $g \in S^1$, $(x_1, \ldots, x_{n+1}, z_1, \ldots, z_k) \in S^{n+2k}$, the action is defined by

$$g(x_1, \ldots, x_{n+1}, z_1, \ldots, z_k) = (x_1, \ldots, x_{n+1}, g z_1, \ldots, g z_k).$$

We denote this action simply by $(S^1, S^{n+2k}, S^n)$.

Let $(S^1, \Sigma^{n+2k}, \Sigma^n)$ be any semifree differentiable action of $S^1$ on $\Sigma^{n+2k}$ with untwisted fixed point set $\Sigma^n$. Then the action around the fixed point set is equivalent to $(S^1, \Sigma^n \times D^{2k}, \Sigma^n \times 0)$ given by

$$g(x, z_1, \ldots, z_k) = (x, g z_1, \ldots, g z_k),$$

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for all \( g \in \mathcal{S}^1 \), \( x \in \Sigma^n \) and \((z_1, \cdots, z_k) \in \mathbb{D}^k\), where \( g z_i \) \((i = 1, \cdots, k)\) is the complex multiplication of \( g \) and \( z_i \) in \( C \). Let \(-\Sigma^n\) be the homotopy sphere \( \Sigma^n \) with orientation reversed and \((\mathcal{S}^1, -\Sigma^{n+2k}, -\Sigma^n)\) be the action induced by \((\mathcal{S}^1, \Sigma^{n+2k}, \Sigma^n)\). Let \((\mathcal{S}^1, \Sigma_1^{n+2k}, \Sigma_1^n)\) and \((\mathcal{S}^1, \Sigma_2^{n+2k}, \Sigma_2^n)\) be any two semifree actions with untwisted fixed point sets. Since the actions around the fixed point sets \( \Sigma_1^n \) and \( \Sigma_2^n \) are equivalent, the equivariant connected sum \((\mathcal{S}^1, \Sigma_1^{n+2k} \# \Sigma_2^{n+2k}, \Sigma_1^n \# \Sigma_2^n)\) is well defined. Two semifree \( \mathcal{S}^1 \) actions with untwisted fixed point sets, \((\mathcal{S}^1, \Sigma_1^{n+2k}, \Sigma_1^n)\) and \((\mathcal{S}^1, \Sigma_2^{n+2k}, \Sigma_2^n)\) are said to be equivalent if the underlying knots \((\Sigma_1^{n+2k}, \Sigma_1^n)\) and \((\Sigma_2^{n+2k}, \Sigma_2^n)\) are isotopic. The equivalence class of \((\mathcal{S}^1, \Sigma^{n+2k}, \Sigma^n)\) is denoted by \([\mathcal{S}^1, \Sigma^{n+2k}, \Sigma^n]\). Let \(SF(n+2k, n)\) be the set of equivalent classes. Then \(SF(n+2k, n)\) is an abelian group under the equivariant connected sum operation:

\[
[S^1, \Sigma_1^{n+2k}, \Sigma_1^n] + [S^1, \Sigma_2^{n+2k}, \Sigma_2^n] = [S^1, \Sigma_1^{n+2k} \# \Sigma_2^{n+2k}, \Sigma_1^n \# \Sigma_2^n].
\]

For if \((\mathcal{S}^1, \Sigma^{n+2k}, \Sigma^n)\) is a representative of \([S^1, \Sigma^{n+2k}, \Sigma^n]\) in \(SF(n+2k, n)\), then the imbedding \(\Sigma^n \# -\Sigma^n \rightarrow \Sigma^{n+2k} \# -\Sigma^{n+2k}\) is isotopic to the standard imbedding \(S^n \rightarrow S^{n+2k}\). Hence \(SF(n+2k, n)\) is an abelian group with zero element \([\mathcal{S}^1, \Sigma^{n+2k}, \Sigma^n]\).

Let \(SF(n+2k, n)^\ast\) and \(SF(n+2k, n)^{**}\) be the subgroups of \(SF(n+2k, n)\) consisting of elements \([S^1, \Sigma^{n+2k}, \Sigma^n]\) with \(\Sigma^{n+2k} = \Sigma^{n+2k}\) and \(\Sigma^n = \Sigma^n\) respectively. Define

\[
SF(n+2k, n)^\sim = SF(n+2k, n)^\ast \cap SF(n+2k, n)^{**}.
\]

Let us recall that \(\theta^{n+2k,n}\) denotes the group of isotopy classes of knotted \(n\)-spheres in \(S^{n+2k}\) [6], and \(bP_n\) as be in [4, p. 510].

**Definition 2.** We define the homomorphisms

\[
\alpha(n+2k,n) : SF(n+2k,n) \rightarrow \theta_{n+2k}
\]

and

\[
\beta(n+2k,n) : SF(n+2k,n) \rightarrow \theta_n
\]

by

\[
\alpha(n+2k,n)[S^1, \Sigma^{n+2k}, \Sigma^n] = \Sigma^{n+2k}
\]

and

\[
\beta(n+2k,n)[S^1, \Sigma^{n+2k}, \Sigma^n] = \Sigma^n.
\]

The diagram below is clearly commutative with exact rows and columns:
Lemma 1. The groups \( SF(n+2k, n)^- \), \( SF(n+2k, n)^* \), \( SF(n+2k, n)^*^* \) and \( SF(n+2k, n) \) are infinite if \( n \equiv 3 \pmod{4} \) and \( 4k-1 \leq n \).

Proof. Let \( \Sigma^{n+2k} \) be the kernel of \( \theta^{n+2k} : \theta_n \), and let \( \Sigma_0^{n+2k} \) be the subgroup of \( \Sigma^{n+2k} \) of knotted spheres which bound framed submanifolds in \( S^{n+2k} \). Then \( \Sigma_0^{n+2k} \approx \mathbb{Z} \) under the hypotheses by [6]. According to [7, 5.2, 5.4], there exist infinitely many elements in \( \Sigma_0^{n+2k} \) such that every element \( [S^{n+2k}, S^n] \) has a representative \( (S^{n+2k}, S^n) \) which can be realized as the fixed point knot of a semifree differentiable \( S^1 \) action on \( S^{n+2k} \) with untwisted fixed point set \( S^n \).

Thus \( SF(n+2k, n)^- \) is infinite. This proves Lemma 1.

To prove Theorem 1, let \( \Sigma^{n+2k} \subseteq \text{Im} \alpha(n+2k, n) \) and \( \beta \subseteq \text{Im} \beta(n+2k, n) \). Then \( \alpha(n+2k, n)^{-1}(\Sigma^{n+2k}) \cap \beta(n+2k, n)^{-1}(\Sigma^n) \) contains infinitely many elements by Lemma 1. The proof is complete.

Now we recall a theorem of Browder [2, (6.2)]:

Theorem 3 (Browder). Suppose \( n \equiv 3 \pmod{4} \) and \( k > 1 \), \( k \) odd, and let \( I_0(CP^{k-1} 	imes S^n) = \{ \Sigma \in \theta_{n+2k-2} \mid (CP^{k-1} \times S^n) \# \Sigma \) is diffeomorphic to \( CP^{k-1} \times S^n \} \cap bP_{n+2k-1} \) and \( l = \text{order of } I_0(CP^{k-1} \times S^n) \).

Then an element \( \Sigma \in bP_{n+1} \), \( n > 3 \), occurs as an untwisted fixed point set of a semifree \( S^1 \) action on a homotopy sphere \( \Sigma^{n+2k} \) if and only if \( \Sigma \in (m_{n,k}/l)bP_n \), where \( m_{n,k} \) is the order of \( bP_{n+2k-1} \).

Applying Theorem 3 to \( n = 11 \) and \( k = 3 \), since the orders of \( \theta_{17} \) and \( bP_{14} \) are 16 and 8128 respectively, we may use the connected sum method to show that there are semifree \( S^1 \) actions on \( S^{17} \) with every element of \( 8128\theta_{11} = 32\theta_{11} = Z_{11} \) as untwisted fixed point set. But \( 4k-1 = n \), so we can apply Theorem 1. This completes the proof of Theorem 2.

Browder has found some exotic spheres in \( \text{Im} \beta(n+2k, n) \) [2]. In general the groups \( \text{Im} \alpha(n+2k, n) \) and \( \text{Im} \beta(n+2k, n) \) are hard to compute.
References


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