A REMARK ON THE DENSITY CHARACTER OF FUNCTION SPACES

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Concerning the density character of function spaces, the most general theorem we know is the unique theorem in [4]—whose brilliant proof carries over verbatim to arbitrary infinite cardinals—and its generalization in [5, (J) and (D)]. The present note investigates the case of the range space separable without countable base as in Michael's theorem [4]. A positive result is given for the range a convex subset of a locally convex space, or an injective uniform space.

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Notations. We shall denote by
- $C(X, Y)$ the set of all continuous maps $X \to Y$.
- $C_c(X, Y)$ the set $C(X, Y)$ with the compact-open topology.
- $C_p(X, Y)$ the set $C(X, Y)$ with the product topology.
- $w(X)$ the weight of $X = \text{least cardinal of a basis for the topology of } X$.
- $dc(X)$ the density character of $X = \text{least cardinal of a dense subset of } X$.

Theorem. Let $X$ be a topological space and $Y$ a convex subset of a locally convex space $Z$. If $Y$ is equipped with the induced topology, then

$$dc(C_c(X, Y)) \leq w(X) \cdot dc(Y)$$

provided that $w(X)$ or $dc(Y)$ is infinite.

Proof. Because a translation is a homeomorphism preserving convexity, we may assume—as we do—that $0 \in Y$. Let $I^n$ be the product of the unit interval $n$ times ($n \in \mathbb{Z}^+$), and let $J_n$ be the subset of $I^n$ defined by the condition: $x \in J_n$ iff the sum of the coordinates of $x$ is less than or equal to 1. Because $Y$ is convex and $0 \in Y$, there is a natural map $\Phi_n$: $C_c(I^n, J_n) \times Y^n \to C_c(I^n, Y)$ defined by

$$\Phi_n(f, y) = \left( x \mapsto \sum_{i=1}^{n} f_i(x) y_i \right)$$

where $f = (f_1, \cdots, f_n) \in C(I^n, J_n)$ and $y = (y_1, \cdots, y_n) \in Y^n$. Clearly $\Phi_n$ is continuous. From the fact that compact $T_1$-spaces (as all compact subsets of $X$ are under the weak topology determined by an $f \in C(X, Y)$) admit partitions of unity and from the fact that $Y$ is AE(metric)—hence AE(pseudometric)—by Dugundji extension theo-

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rem [2, ix.6.1], it is easily seen that $\bigcup_{n=1}^{\infty} \Phi_n(C_c(X, J_n) \times Y^n)$ is dense in $C_c(X, Y)$. By the same proof of the unique theorem in [4] we may establish the following theorem:

If $\aleph$ is an infinite cardinal and $X$, $Y$ are arbitrary topological spaces with $w(X)$, $w(Y)$ $\leq \aleph$, then hereditarily $dc(C_c(X, Y)) \leq \aleph$.

This theorem implies that $dc(C_c(X, J_n)) \leq w(X) \cdot \aleph_0$ because $J_n$ is separable metric. Moreover, $dc(Y^n) \leq dc(Y) \cdot \aleph_0$. Therefore, by the continuity of $\Phi_n$,

$$dc(\Phi_n(C_c(X, J_n) \times Y^n)) \leq w(X) \cdot dc(Y) \cdot \aleph_0^2 = w(X) \cdot dc(Y).$$

Hence the result. Q.E.D.

The above result is poor in the case of $X$ discrete: If $\text{Card}(X) = 2^{\aleph_0}$, then $C_c(X, \mathbb{R})$ is separable by [2, viii.7.2.(3)], although the above theorem only gives a dense subset of cardinality $\leq 2^{\aleph_0}$. But the above theorem is best possible in case $X$ is compact: If $X$ is the product of the unit interval $2^{\aleph_0}$ times, then $X$ is compact, separable (by [2, viii.7.2.(3)]) and nonmetrizable; therefore $w(X) = 2^{\aleph_0}$, and $C_c(X, \mathbb{R})$ cannot be separable, because otherwise $X$ would be metrizable by [1, §3, Theorem 1].

The property of the above theorem is not hereditary: Let $K$ be the Cantor space. Because $K$ is uncountable, $C_p(K, C_p(K, [0, 2]))$ is not hereditarily separable as the proof of [5, Lemma 10.6] indicates. And $C_p(K, [0, 2]))$ is a separable convex subset of a locally convex space—the product $\mathbb{R}^K$.

A natural question: Is the above theorem true for general $Y$? A positive answer can be derived as a corollary of the above theorem via a uniform embedding into a product of Banach spaces when $Y$ is an injective uniform space for some uniformity (according to [3, p. 39], a uniform space $Y$ is called injective iff, whenever $A$ is a uniform subspace of $X$, every uniformly continuous map $A \to Y$ can be extended to a uniformly continuous map $X \to Y$).

References


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