NOTE ON RELATIVE $p$-BASES OF PURELY INSEPARABLE EXTENSIONS

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Throughout this note $L/K$ denotes a purely inseparable field extension of characteristic $p$ and nonzero exponent. In [5, p. 745], Rygg proves that when $L/K$ has bounded exponent, then a subset $M$ of $L$ is a relative $p$-base of $L/K$ if and only if $M$ is a minimal generating set of $L/K$. The purpose of this note is to answer the following question: If every relative $p$-base of $L/K$ is a minimal generating set, then must $L/K$ be of bounded exponent? The answer is known to be yes when $K$ and $L'$ are linearly disjoint, $i = 1, 2, \cdots$, see [1]. We give two examples for which the answer is no: One in which the maximal perfect subfield of $L$ is contained in $K$, and the other in which it is not.

The following lemmas are needed for our examples. An intermediate field $L'$ of $L/K$ is called proper if $K \subseteq L' \subset L$.

**Lemma 1.** Every relative $p$-base of $L/K$ ([2, p. 180]) is a minimal generating set of $L/K$ if and only if there does not exist a proper intermediate field $L'$ of $L/K$ such that $L = L'(L^p)$.

**Proof.** If $L = L'(L^p)$, where $L'$ is a proper intermediate field of $L/K$, then $L'$ contains a relative $p$-base $M$ of $L/K$. Thus $L' \supseteq K(M)$. Conversely, if there exists a relative $p$-base $M$ of $L/K$ such that $L \supseteq K(M)$, then $L = L'(L^p)$, where $L' = K(M)$. Q.E.D.

**Lemma 2.** Suppose $L = K(m_1, m_2, \cdots)$, where $m_i \in K(m_{i+1})$, $i = 1, 2, \cdots$. Then $K, K(m_i), L$ are the intermediate fields of $L/K$, $0 \leq j_i < e_i$ (the exponent of $m_i$ over $K(m_{i-1})$), $i = 1, 2, \cdots$, where $K(m_0)$ means $K$.

**Proof.** Let $e_i$ denote the exponent of $m_i$ over $K$, $i = 1, 2, \cdots$. By [2, p. 196, Exercise 5], the intermediate fields of $K(m_i)/K$ are $K(m_i^{e_i})$, $0 \leq j_i \leq e_i$. If $0 < i < s$, then $K(m_i) \subseteq K(m_s)$, whence $K(m_i) = K(m_i^{e_i^{i,s}})$. Thus the intermediate fields of $K(m_i)/K$ are $K, K(m_i^{e_i})$, $0 \leq j_i < e_i$, $i = 1, \cdots, s$. Let $K'$ be any intermediate field of $L/K$. If $[K':K] < \infty$, then $K'/K$ is finitely generated. Hence $K' \subseteq K(m_s)$ for some $m_s$ since $L = \bigcup_{i=1}^{s} K(m_i)$. Thus $K' = K(m_i^{e_i})$ for some $m_i$ by the preceding argument. If $[K':K] = \infty$, then $K'$ is the union over $c$ of $K(c)$ for all $c \in K'$. Now $K(c) = K(m_i^{e_i}) \supseteq K(m_{i-1})$ for some $m_i$.
and for all \( c \in K' - K \) by the previous argument. Since \( [K':K] = \infty \), \( i_c \) is an unbounded function of \( c \). Thus \( K' = L \). Q.E.D.

**Example 1.** \( L/K \) is of unbounded exponent, the maximal perfect subfield of \( L \) is not contained in \( K \), and every relative \( p \)-base of \( L/K \) is a minimal generating set of \( L/K \): Let \( P \) be a perfect field and \( z, y, x_1, x_2, \cdots \) independent indeterminates over \( P \). Let \( K = P(z, y, x_1, x_2, \cdots) \) and \( L = K(m_1, m_2, \cdots) \), where \( m_i = z^{p-i-1} x_i^{p-1} + y^{p-1} \), \( i = 1, 2, \cdots \). Clearly, \( L/K \) is of unbounded exponent. \( P(z^{p-1}, y^{p-2}, \cdots) \) is the maximal perfect subfield of \( L \) and is not in \( K \). By Lemma 1, every relative \( p \)-base of \( L/K \) is a minimal generating set of \( L/K \) if we show that \( L \notin L'(L^p) \) for any proper intermediate field \( L' \) of \( L/K \). We postpone this proof.

**Example 2.** \( L/K \) is of unbounded exponent, the maximal perfect subfield of \( L \) is contained in \( K \), and every relative \( p \)-base of \( L/K \) is a minimal generating set of \( L/K \): Let \( P \) be a perfect field and \( z, y, x_1, x_2, \cdots \) independent indeterminates over \( P \). Let \( K = P(y, x_1, x_2, \cdots) \) and \( L = K(m_1, m_2, \cdots) \), where \( m_1 = x_1^{p-2} \) and \( m_{i+1} = (m_i^p y + x_{i+1})^{p-2} \), \( i = 1, 2, \cdots \). Clearly, \( L/K \) is of unbounded exponent. It follows that \( L = P(y, m_1, m_2, \cdots) \) and that \( \{ y, m_1, m_2, \cdots \} \) is an algebraically independent set over \( P \). That is, \( L/P \) is a pure transcendental extension. Thus \( P, P \subseteq K \), is the maximal perfect subfield of \( L \) by Corollary 2 of \([3, p. 388]\). By Lemma 1, it remains to be shown that \( L \neq L'(L^p) \) for any proper intermediate field \( L' \) of \( L/K \).

We prove simultaneously for Examples 1 and 2 that such a field \( L' \) cannot exist. In both examples, it follows that \( K(L^p) = K(m_1^p, m_2^p, \cdots) \), \( m_i^p \subseteq K(m_{i+1}^p) \) and \( m_{i+1}^p \) has exponent 1 over \( K(m_i^p) \) for \( i = 1, 2, \cdots \). Hence, by Lemma 2, the intermediate fields of \( K(L^p)/K \) appear in a chain. Now suppose there exists a proper intermediate field \( L' \) of \( L/K \) such that \( L = L'(L^p) \). Since \( L' \cap K(L^p) = K(m_i^p) \) for some integer \( s \geq 0 \). We show \( L' \) and \( K(L^p) \) are linearly disjoint over \( K(m_i^p) \) by showing that for every proper intermediate field \( K' \) of \( K(L^p)/K(m_i^p) \), \( L' \) and \( K' \) are linearly disjoint over \( K(m_i^p) \). By Lemma 2, \( K' = K(m_i^p) \) for \( t \geq s \). Now \( m_i^p \) has exponent \( t-s \) over \( K(m_i^p) \subseteq L' \). If \( ((m_i^p)^{p^r})^{p^r} \subseteq L' \), then we contradict \( L' \cap K(L^p) = K(m_i^p) \). Hence the irreducible polynomial of \( m_i^p \) over \( K(m_i^p) \) remains irreducible over \( L' \). Thus \( L' \) and \( K(m_i^p) \) are linearly disjoint over \( K(m_i^p) \), whence \( L' \) and \( K(L^p) \) are linearly disjoint.

Since \( L = L'(L^p) \), \( m_{s+1} \in L' \cap L(p) \). Hence \( m_{s+1} = \sum_{j=0}^{e-1} c_j \cdot (m_i^p)^j \) for some integer \( t \), where \( c_j \in L' \). Now \( t \geq s+2 \) since \( m_{s+1} \) has exponent 2 over \( K(m_i^p) \) and \( L' \) has exponent 1 over \( K(m_i^p) \). Thus

\[
m_{s+1}^p = \sum_j c_j (m_i^p)^{ip}.
\]
By the division algorithm, \( j p = p^{i-r} q_i + r_i \), \( 0 \leq r_i < p^{i-r} \). Hence

\[
(*) \quad m_{i+1}^p = \sum_i \left( c_i^p (m_i^p) p^{i-r} q_i \right)(m_i^p)
\]

and

\[
c_i^p (m_i^p) p^{i-r} q_i = c_i^p \in L' \cap L^p.
\]

Writing \( m_{i+1}^p \) in terms of \( m_i^p \), we get for Example 1:

\[
m_{i+1}^p = (m_i^p) p^{i-r+1} k_0 x_{i+1} - k_1,
\]

where

\[
k_0 = x_i^{-p^{i-r+1}}
\]

and

\[
k_1 = y x_i^{-p^{i-r+1}} x_{i+1} - y.
\]

By (*)

\[
(m_i^p) p^{i-r+1} x_{i+1} = k_0 x_i^{-p^{i-r+1}} + k_0 x_i \sum c_i^p (m_i^p)^{i-r}.
\]

Hence, by the linear disjointness of \( L' \) and \( K(L^p) \) over \( K(m_i^p) \) and since \( \{ (m_i^p)^j \mid j = 0, \ldots, p^{i-r} - 1 \} \) is linearly independent over \( K(m_i^p) \), \( x_{i+1} \in L' \cap L^p \). Thus \( x_{i+1} \in L \), a contradiction.

For Example 2, we get

\[
m_{i+1}^p = (m_{i+2}^p) y^{-1} - x_{i+2} y^{-1} = \cdots = (m_i^p) p^{i-r+1} k_0 y^{-1} - k_1
\]

for suitable \( k_0, k_1 \in K \). By an argument similar to that in Example 1, we obtain \( y x_{i+1} \in L \), a contradiction.

REMARK. Let \( P \) denote the maximal perfect subfield of \( L \) and \( M \) a relative \( p \)-base of \( L/K \). Consider the properties: (1) \( P \subseteq K \), (2) \( P \not\subseteq K \), (3) there exists an \( M \) such that \( L = K(M) \), and (4) for all \( M, L = K(M) \). None implies the other except for (4) implies (3).

For instance, Example 1 shows that (4) \( \Rightarrow \) (1) and Example 2 shows that (4) \( \Rightarrow \) (2). Example 2 of \( [4, \text{p. 333}] \) shows that (3) \( \Rightarrow \) (4).

Letting \( L \) be perfect gives us an example showing that (2) \( \Rightarrow \) (3). We show (1) \( \Rightarrow \) (3) by giving an example constructed by E. A. Hamann. Let \( K = Q(x_1, x_2, \ldots) \) and

\[
L = K(x_1^{p-1}, (x_1 + x_2)^{p-2}, (x_1 + x_2 + x_1^2)^{p-3}, \ldots),
\]

where \( Q \) is a perfect field and \( x_1, x_2, \ldots \) are independent indeterminates over \( Q \). \( L = K(L^p) \) and \( L/Q \) is pure transcendental. The remaining implications are trivial.
References


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