EXTENSIONS OF GROUP REPRESENTATIONS OVER FIELDS OF PRIME CHARACTERISTIC

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Let G be a finite group having a normal Hall subgroup H, let K be a field, and let T be an irreducible (linear) K-representation of H of degree deg T whose character is invariant under the action of G. We say that T is extendible to G if there exists a K-representation S of G such that $S(h) = T(h)$ for all $h \in H$. In [**5**, Theorem 6] Gallagher proved that T is extendible if K is the field of complex numbers. The case when K is an arbitrary field of characteristic zero is treated by Isaacs in [**7**]. In this note we show that the arguments in Isaacs' paper can be extended to yield the following result:

THEOREM. *If K has characteristic p, $p > 0$, and if either H is p-solvable or* (deg T, $[G: H]$) $= 1$, *then T is extendible to G.*

The following lemma is essentially a corollary of the Swan-Fong theorem [**8**, Theorem 6]. An elementary proof of this result appears in [**2**].

LEMMA 1. *Let D denote the set of all finite groups B with the property that if H is a subgroup of B, then the degrees of the absolutely irreducible linear and absolutely irreducible projective K-representations of H divide the order $|H|$ of H. If G is a finite group such that every composition factor of G lies in D, then the degrees of the absolutely irreducible linear K-representations of G divide $|G|$. In particular, if K has characteristic p and G is p-solvable, the degrees of the absolutely irreducible linear K-representations of G divide $|G|$.*

PROOF. The first part of the lemma is easily proved by induction using the argument of Theorem 53.17 of [**1**]. The statement about p-solvable groups now follows from [**1**, Theorems 27.28 and 53.3] together with the remarks in [**1**, p. 600].

LEMMA 2. *Let W be an absolutely irreducible representation of H which is a constituent of T. If* (deg T, $[G: H]$) $= 1$, *then* (deg W, $[G: H]$) $= 1$.

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PROOF. This follows immediately from the theory of the Schur index [3, Theorem 1.4].

If V is a representation of a group B we denote by det V the one-dimensional representation of B obtained by taking the determinant of V. det V is an element of the group of one-dimensional representations of B and we denote by $o(V)$ the order of det V in this group.

LEMMA 3. *Let K be algebraically closed of characteristic p and suppose that* $(\deg T, [G: H])=1$. *Then there is a unique extension S of T to G such that* $(o(S), [G: H])=1$.

PROOF. The fact that T is extendible to G follows from the proof of Satz 17.12 on p. 572 in [6]. Let U be some extension of T to G. The proof of Theorem 51.7 of [1] shows that if S is any extension of T to G then there exists a projective representation Y of G/H such that $S(g)\times U(g)\times Y(g)$ (Kronecker product) for all $g\in G$. Thus Y is a one-dimensional linear representation of G/H. The result now follows from the proof of [5, Theorem 5].

PROOF OF THE THEOREM. Let $E=K(\sqrt[n]{1})$, $n=|G|$. Let W be an irreducible E-representation of H which is a constituent of T and let θ be the character of W. From the theory of the Schur index we have

$$T^E \sim \sum_{\oplus} W^\sigma$$

the sum ranging over all automorphisms σ in the Galois group $\mathcal{G}(K(\theta)|K)$ of $K(\theta)$ over K [3, Theorem 1.4]. Since $K(\theta)\subset E$, $\mathcal{G}(K(\theta)|K)$ is cyclic. Thus I, the inertia group of θ, is a normal subgroup of G, $H\subset I\subset G$ and G/I is cyclic [7, Lemma 1.2]. By Lemmas 1, 2, and 3 there is a unique extension $\hat{W}$ of W to I such that $(o(\hat{W}), [I: H])=1$. Let $V=\hat{W}^G$. V is an irreducible E-representation of G. Let χ be the character of V. Then V is realizable in $K(\chi)$. If $\tau\in\mathcal{G}(K(\chi)|K)$, $\tau\neq 1$, then $V|_H$ and $V^\tau|_H$ have no constituents in common [7, Proposition 1.4]. Let S be an irreducible K-representation of G such that $S^E\sim\sum_\oplus V^\tau$ the sum ranging over all $\tau\in\mathcal{G}(K(\chi)|K)$. It suffices to prove that $S|_H\sim T$. T is a constituent of $S|_H$, so by Clifford's Theorem $S|_H\sim eT$.

The remarks above show that e is also the multiplicity with which W occurs as a constituent of $V|_H$. The multiplicity of $\hat{W}$ in $V|_I$ is 1, $\hat{W}$ is the unique constituent of $V|_I$ whose restriction to H is W. Thus $e=1$, which proves the result.

REMARK. Lemma 3 has an application to the question of determining when an indecomposable K-representation of H is extendible to a K-representation of G. Assume that H is p-solvable and K is algebraically closed of characteristic p, $(p, [G: H])=1$. Let U be a principal indecomposable $K[H]$-module such that $U^{(g)}\cong U$ for all $g\in G$. Let M be the unique minimal submodule of U. Clearly $M^{(g)}\cong M$ for all $g\in G$ so by Lemma 3, M is extendible to a $K[G]$-module N, $N_H=M$. Let V be a principal indecomposable $K[G]$-module having N as its unique minimal submodule. In view of Theorem (2B) of [4], to prove that $V_H=U$ it is sufficient to show that $U|V_H$. Let R be the sum of the irreducible $K[H]$-submodules of V_H. Then $M=N_H\subset R$ so M is a direct summand of R. But V_H is injective and so contains an injective hull of M [1, Theorem 57.13]. Since U is the injective hull of M we have proved that U is extendible to V.

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