AN INFINITE DIMENSIONAL VERSION OF A
THEOREM OF BERNSTEIN

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1. Introduction. Let \( \mathcal{P}(\mathbb{R}^n) \) be the algebra of polynomials in \( n \) variables with the topology of uniform convergence on bounded sets of a function \( f \) and its derivative \( f' \). A classical theorem of Bernstein says that the closure of \( \mathcal{P}(\mathbb{R}^n) \) is the algebra \( \mathcal{C}^1(\mathbb{R}^n) \) of all real-valued functions of class \( C^1 \). In other words, for every \( f \in \mathcal{C}^1(\mathbb{R}^n) \) there is a sequence \( \{ p_n \} \) of polynomials such that \( p_n \to f \) uniformly on bounded sets and \( f'_n \to f' \) uniformly on bounded sets. In this paper we define the algebra \( \mathcal{P}(X) \) of polynomials in a Banach space \( X \) and determine its closure for a restricted class of reflexive Banach spaces (Theorem 8). Thus, Theorem 8 answers a question raised by the author in [2]. In what follows, weak convergence in \( X \) will be denoted by \( x_n \to x \) and strong convergence by \( x_n \to x \). Henceforth we will assume all Banach spaces to be separable.

2. Polynomials in Banach spaces. Let \( X \) be a Banach space and let \( X^* \) be its dual. If \( u = (u_1, \ldots, u_n) \in (X^*)^n \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is a sequence of positive integers, we define \( u^\alpha \) to be the real valued function \( u^\alpha(x) = (u_1(x))^{\alpha_1} \cdots (u_n(x))^{\alpha_n} \) and \( |\alpha| = \alpha_1 + \cdots + \alpha_n \). A polynomial in \( X \) is a function \( p: X \to \mathbb{R} \) of the form \( p = \sum_{|\alpha| \leq m} a_\alpha u^\alpha \) where \( a_\alpha \) is a real number.

Let \( X, Y \) be Banach spaces. An operator \( T: X \to Y \) is weakly continuous if \( T \) is continuous from the weak topology in \( X \) to the norm topology in \( Y \).

**Theorem 1.** Let \( X \) be a Banach space and let \( p \) be a polynomial. Then

(i) \( p \) is weakly continuous.

(ii) The derivative \( p': X \to X^* \) is weakly continuous.

(iii) \( p'(X) \) is finite dimensional.

**Proof.** (i) follows from the definition of \( p \). In order to prove (ii) and (iii) we simply calculate \( p'(x) \). If \( p(x) = u(x)v(x) \), where \( u, v \in X^* \), then \( p'(x) = u(x)v + v(x)u \), and if \( p(x) = (u(x))^k \) then \( p'(x) = k(u(x))^{k-1}u \). Now, by induction, one can show if \( p = u^\alpha = u_1^{\alpha_1} \cdots u_n^{\alpha_n} \) then \( p'(x) \) is of the form \( \sum_{i=1}^n \lambda_i(x)u_i \) where \( \lambda_i \) is a weakly continuous real valued function. In this case \( p' \) satisfies (ii) and (iii). The general case follows from the previous ones.

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Theorem 2. Let $X$, $Y$ be Banach spaces and $f : X \to Y$ continuous. Let $P : X \to Y$ be linear and bounded and such that $P(X)$ is finite dimensional. Then

(i) $f \circ P$ is weakly continuous.

(ii) $(f \circ P)' = P^* \circ f' \circ P$.

Proof. (i) Let $x_n \to x$ ($x_n$ converges weakly to $x$). Then $P x_n \to P x$ since $P$ is linear. Furthermore, $P x_n \to P x$ since $P(X)$ is finite dimensional, so $f P(x_n) \to f P(x)$ since $f$ is continuous.

(ii) A trivial calculation.

Theorem 3. Let $X$ be a reflexive Banach space. Then $f : X \to \mathbb{R}$ is weakly continuous on bounded sets if and only there is a sequence $\{p_n\}$ of polynomials such $p_n \to f$ uniformly on bounded sets.

Proof. Every set of the form $\{x \mid \|x\| \leq M\}$ is weakly compact, so $f$ is weakly continuous if $p_n \to f$ uniformly on bounded sets. Let us prove the second part. The theorem of Stone-Weierstrass can be used to show that for every continuous function $f$ on bounded sets and every integer $n$ there is a polynomial $p_n$ such that $|f(x) - p_n(x)| < 1/n$ for all $x$ such that $\|x\| \leq n$. The sequence $\{p_n\}$ converges to $f$ uniformly on bounded sets. This concludes the proof.

Remark. Weakly continuous on bounded sets means: the restriction of $f$ to any bounded subset of $X$ is relatively weakly continuous. If a function is weakly continuous, then it is weakly continuous on bounded sets. The converse is not true. Let $X = H$ be a Hilbert space, let $e_1, e_2, \ldots, e_n, \ldots$ be a basis of $H$ and let

$$f(x) = \sum_{n=1}^{\infty} \frac{(e_n(x))^n}{(2n)!}.$$  

If this function were weakly continuous at the origin of $H$ then it would be bounded on some subspace $M$ of finite codimension in $H$. For $x \in M$ fixed,

$$g(z) = \sum_{n=1}^{\infty} \frac{(e_n(x))^n}{(2n)!} z^n$$

is an entire function of order $1/2$ in $z$. Such a function cannot be bounded for all real $z$ unless it is a constant, in this case zero. Hence $M = \{0\}$ which is a contradiction. I am grateful to the referee for pointing out this nice example and for other constructive comments.

We say that a Banach space $X$ has property (B) if there is a sequence $P_n : X \to X$ of bounded linear operators such that
(i) \( P_n(X) \) is finite dimensional for each \( n \).

(ii) \( P_n^2 = P_n \).

(iii) \( P_n(x) \to x \) for every \( x \in X \).

(iv) \( P_n^*(u) \to u \) for every \( u \in X^* \).

Every Banach space with a biorthogonal basis has property (B). In particular, every Hilbert space has property (B). The theorem below has been proved by Vainberg in [1] for reflexive Banach spaces with a biorthogonal basis. Our proof is similar to his.

**Theorem 4.** Let \( X \) be a reflexive Banach space with property (B). Let \( T: X \to X^* \) be a continuous operator. Then \( T \) is weakly continuous on bounded sets if and only if \( T \circ P_n \to T \) uniformly on bounded sets.

**Proof.** By the uniform boundness principle there is a constant \( M \) such that \( \|P_n\| \leq M \) and \( \|P_n^*\| \leq M \) for all \( n \). Let us show that if \( x_n \to x \) then \( P_n x_n \to x \). Now, \( \langle P_n x_n, u \rangle = \langle x_n, P_n^* u \rangle = \langle x_n, u \rangle + \langle x_n, P_n^* u - u \rangle \). Since \( P_n^* u \to u \), and \( \{x_n\} \) is bounded, it follows that \( \langle P_n x_n, u \rangle \to 0 \) for every \( u \in X^* \).

Let us assume that \( T \) is weakly continuous on bounded sets and suppose that \( T \circ P_n \) does not converge to \( T \) uniformly on bounded sets. Then, for some \( \varepsilon > 0 \) and for every integer \( k \) there exist an integer \( n_k \) and a point \( x_k \) such that \( \|x_k\| \leq M \) for some \( M > 0 \) and \( \|TP_{n_k} x_k - Tx_k\| \leq \delta, k = 1, 2, \ldots \). Since \( X \) is reflexive, there is a subsequence (still denoted by \( \{x_k\} \)) such that \( x_k \to x_0 \) for some \( x_0 \). This would imply that \( P_{n_k} x_k \to x_0 \) and \( TP_{n_k} x_k \to Tx_0 \). Thus, we have reached a contradiction.

Let us assume that \( T \circ P_n \to T \) uniformly on bounded sets. Since \( T \circ P_n \) is weakly continuous by Theorem 2 and \( \{x \mid \|x\| \leq M\} \) is weakly compact for every \( M > 0 \), it follows that \( T \) is weakly continuous on bounded sets.

**3. Uniformly differentiable functions.** Let \( X \) and \( Y \) be Banach spaces and let \( f: X \to Y \) be of class \( C^1 \). We say that \( f \) is uniformly differentiable in a subset \( A \subset X \) if for every \( \varepsilon > 0 \) there is some \( \delta > 0 \) such that

\[
\|f(a + h) - f(a) - \langle f'(a), h \rangle\| \leq \varepsilon \|h\|
\]

whenever \( \|h\| \leq \delta \) for all \( a \in A \).

**Theorem 5.** Let \( X \) be a reflexive Banach space with property (B). Let \( f: X \to \mathbb{R} \) be weakly continuous and uniformly differentiable on bounded sets. Let \( T = f': X \to X^* \). Then \( P_n^* \circ T \to T \) uniformly on bounded sets.

**Proof.** Let \( I: X \to X \) be the identity operator and let \( h = (I - P_n)(x) \).
Let \( O(x, h) = f(x + h) - f(x) - \langle T(x), h \rangle \). Since \( \langle Tx, h \rangle = \langle (I - P_n^*) Tx, h \rangle \) we get,

\[
\left| \langle (I - P_n^*)Tx, h \rangle \right| \leq \left| f(x + h) - f(x) \right| + \left| O(x, h) \right|
\]

(1)

\[
\leq \left| f(x + h) - f(P_n(x + h)) \right| + \left| f(P_n(x + h) - f(x) \right| + \left| O(x, h) \right|
\]

Let \( \epsilon > 0 \) be given and let \( M > 0 \). Since \( f \) is uniformly differentiable on bounded sets there is some \( \delta > 0 \) such that

\[
\left| O(x, h) \right| < \left( \epsilon / 3 \right) ||h|| \quad \text{if } ||h|| < \delta
\]

(2) for all \( x \) such that \( ||x|| \leq M \). Now, by the uniform boundness principle there is some \( N \) such that \( ||I - P_n|| \leq N \) for all \( n \). Since \( f \) is weakly continuous, there is some \( n_0 \) (see Theorem 4) such that

\[
\left| f(x + h) - f(P_n(x + h)) \right| < \epsilon \delta / 3N
\]

(3) \( \left| f(x) - f(P_n x) \right| < \epsilon \delta / 3N \) for all \( x \) such that \( ||x|| \leq N \) and all \( h \) such that \( ||h|| < \delta \). Let \( y \in X \) such that \( ||y|| = 1 \) and let \( h = (\delta / N)(I - P_n)y \). then \( ||h|| \leq \delta \) and

\[
\left| \langle (I - P_n^*)Tx, y \rangle \right| = \left| \langle (I - P_n^*)Tx, (I - P_n)y \rangle \right|
\]

\[
= \left( M / \delta \right) \left| \langle (I - P_n^*)Tx, h \rangle \right|
\]

(4) from (1), (2), (3) and (4) we get

\[
\left| \langle (I - P_n^*)Tx, y \rangle \right| < \epsilon \quad n \geq n_0
\]

for all \( x \) such that \( ||x|| \leq M \). This concludes the proof.

**Theorem 6.** Let \( X \) be a reflexive Banach. Assume \( f: X \rightarrow \mathbb{R} \) is weakly continuous and uniformly differentiable on bounded sets. Then \( f' = T: X \rightarrow X^* \) is weakly continuous.

**Proof.** We have to show that \( x_n \rightarrow x \) implies \( Tx_n \rightarrow Tx \). Since \( \{x_n\} \) is bounded, we can assume that everything takes place in some bounded set \( B = \{x \mid ||x|| \leq M\} \). Let \( \epsilon > 0 \). By uniform differentiability we can make

\[
\left| O(x, h) \right| < 4^{-1} \epsilon ||h||
\]

(1) if \( ||h|| \leq \delta < 1 \) for some \( \delta \) and all \( x \in B \). Since \( f \) is weakly uniformly continuous in \( 2B \), there is a weak neighborhood \( V(0) \) such that,

\[
\left| f(y) - f(z) \right| < 4^{-1} \delta \epsilon
\]

(2) for all \( y, z \in 2B \) such that \( y - z \in V(0) \). Now,
\[(Tx_n, h) = f(x_n + h) - f(x_n) - O(x_n, h)\]

\[(Tx, h) = f(x + h) - f(x) - O(x, h)\]

and

\[
| \langle Tx_n - Tx, h \rangle | \leq | f(x_n + h) - f(x + h) | + | f(x_n) - f(x) | + | O(x_n, h) | + | O(x, h) | .
\]

Let \(n_0\) be such that \(x_n - x_0 \in V\) if \(n \geq n_0\). From (1), (2) and (3) we obtain

\[
| \langle Tx_n - Tx, h \rangle | < \varepsilon \delta
\]

if \(n \geq n_0\) and \(|h| = \delta\). This shows that \(| \langle Tx_n - Tx, h \rangle | < \varepsilon\) if \(n \geq n_0\) and \(|h| = 1\). Therefore, \(| Tx_n - Tx \| < \varepsilon\) if \(n \geq n_0\). This concludes the proof.

**Theorem 7.** Let \(X\) be a reflexive Banach with property (B). Let \(f : X \to \mathbb{R}\) be weakly continuous and uniformly differentiable on bounded sets. Then \(P_n TP_n \to T\) uniformly on bounded sets.

**Proof.** We have for any \(x \in X\)

\[
\| Tx - P_n TP_n (x) \| \leq \| Tx - P_n T x \| + \| P_n T x - P_n TP_n x \| .
\]

The first term in the right-hand side converges to zero uniformly on bounded sets (Theorem 5). The second term is lesser or equal than \(\| P_n \| \| Tx - TP_n x \|\) which converges to zero uniformly on bounded sets by Theorems 6 and 4 and the fact that \(\| P_n \| \leq M\) for some constant \(M\) and all \(n\). This concludes the proof.

4. **The main theorem.**

**Theorem 8.** Let \(X\) be a reflexive Banach space with property (B). Let \(\mathcal{P}(X)\) be the algebra of polynomials in \(X\) with the topology of uniform convergence on bounded sets of a function and its derivative. Then the closure of \((X)\) is the algebra of weakly continuous functions on bounded sets which are uniformly differentiable on bounded sets.

**Proof.** Let \(f\) be weakly continuous and uniformly differentiable on bounded sets, and let \(f_n = f \circ P_n\). Then \(f_n' = P_n^* f' \circ P_n\) (Theorem 2), \(f_n \to f\) uniformly on bounded sets (Theorem 4) and \(f_n' \to f'\) uniformly on bounded sets (Theorem 7). Therefore, we can assume that that \(f = g \circ P\) where \(P\) is a projection into some finite-dimensional subspace and \(g\) is uniformly differentiable on bounded sets. Let \(P(X) = H\) and let \(h\) be the restriction of \(g\) to \(H\), that is, \(h(x) = g(x)\) for all \(x \in H\). Since \(H\) is finite dimensional there is a sequence \(\{q_n\}\) of
polynomials in $H$ such that $q_n \to h$, $q'_n \to h'$ uniformly on bounded sets (this follows from a classical theorem of Bernstein). Let $p_n = q_n \circ P$. Then,

$$|f(x) - p_n(x)| = |gP(x) - q_n P(x)| \to 0$$

uniformly on bounded sets. Now, let $u \in X$ such that $||u|| = 1$. Then,

$$|\langle f'(x) - p'_n(x), u \rangle| = |\langle g'P(x) - q'_n P(x), Pu \rangle|$$

$$= |\langle h'P(x) - q'_n P(x), Pu \rangle|$$

$$\leq ||P|| \|h'P(x) - q'_n P(x)\|.$$ 

Therefore, $||f'(x) - P_n'(x)|| \leq ||P|| \|h'P(x) - q'_n P(x)\| \to 0$ uniformly on bounded sets. This concludes the proof.

**References**


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