AN INFINITE DIMENSIONAL VERSION OF A THEOREM OF BERNSTEIN

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1. Introduction. Let \( \mathcal{P}(\mathbb{R}^n) \) be the algebra of polynomials in \( n \) variables with the topology of uniform convergence on bounded sets of a function \( f \) and its derivative \( f' \). A classical theorem of Bernstein says that the closure of \( \mathcal{P}(\mathbb{R}^n) \) is the algebra \( \mathcal{C}^1(\mathbb{R}^n) \) of all real-valued functions of class \( C^1 \). In other words, for every \( f \in \mathcal{C}^1(\mathbb{R}^n) \) there is a sequence \( \{ p_n \} \) of polynomials such that \( p_n \to f \) uniformly on bounded sets and \( f'_n \to f' \) uniformly on bounded sets. In this paper we define the algebra \( \mathcal{P}(X) \) of polynomials in a Banach space \( X \) and determine its closure for a restricted class of reflexive Banach spaces (Theorem 8). Thus, Theorem 8 answers a question raised by the author in [2]. In what follows, weak convergence in \( X \) will be denoted by \( x_n \to x \) and strong convergence by \( x_n \to x \). Henceforth we will assume all Banach spaces to be separable.

2. Polynomials in Banach spaces. Let \( X \) be a Banach space and let \( X^* \) be its dual. If \( u = (u_1, \ldots, u_n) \in (X^*)^n \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is a sequence of positive integers, we define \( u^\alpha \) to be the real valued function \( u^\alpha(x) = (u_1(x))^{\alpha_1} \cdots (u_n(x))^{\alpha_n} \) and \( |\alpha| = \alpha_1 + \cdots + \alpha_n \). A polynomial in \( X \) is a function \( p: X \to \mathbb{R} \) of the form \( p = \sum_{|\alpha| \leq m} a_\alpha u^\alpha \) where \( a_\alpha \) is a real number.

Let \( X, Y \) be Banach spaces. An operator \( T: X \to Y \) is weakly continuous if \( T \) is continuous from the weak topology in \( X \) to the norm topology in \( Y \).

Theorem 1. Let \( X \) be a Banach space and let \( p \) be a polynomial. Then

(i) \( p \) is weakly continuous.
(ii) The derivative \( p': X \to X^* \) is weakly continuous.
(iii) \( p'(X) \) is finite dimensional.

Proof. (i) follows from the definition of \( p \). In order to prove (ii) and (iii) we simply calculate \( p'(x) \). If \( p(x) = u(x)v(x) \), where \( u, v \in X^* \), then \( p'(x) = u(x)v + v(x)u \), and if \( p(x) = (u(x))^k \) then \( p'(x) = k(u(x))^{k-1}u \).

Now, by induction, one can show if \( p = u^\alpha = u_1^{\alpha_1} \cdots u_n^{\alpha_n} \) then \( p'(x) \) is of the form \( \sum_{i=1}^n \lambda_i(x)u_i \) where \( \lambda_i \) is a weakly continuous real valued function. In this case \( p' \) satisfies (ii) and (iii). The general case follows from the previous ones.

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Theorem 2. Let $X$, $Y$ be Banach spaces and $f: X \to Y$ continuous. Let $P: X \to Y$ be linear and bounded and such that $P(X)$ is finite dimensional. Then

(i) $f \circ P$ is weakly continuous.

(ii) $(f \circ P)' = P^* \circ f' \circ P$.

Proof. (i) Let $x^\prime \to x$ ($x^\prime$ converges weakly to $x$). Then $Px_n \to Px$ since $P$ is linear. Furthermore, $Px_n \to Px$ since $P(X)$ is finite dimensional, so $fP(x_n) \to fP(x)$ since $f$ is continuous.

(ii) A trivial calculation.

Theorem 3. Let $X$ be a reflexive Banach space. Then $f: X \to \mathbb{R}$ is weakly continuous on bounded sets if and only there is a sequence $\{p_n\}$ of polynomials such $p_n \to f$ uniformly on bounded sets.

Proof. Every set of the form $\{x \mid \|x\| \leq M\}$ is weakly compact, so $f$ is weakly continuous if $p_n \to f$ uniformly on bounded sets. Let us prove the second part. The theorem of Stone-Weierstrass can be used to show that for every continuous function $f$ on bounded sets and every integer $n$ there is a polynomial $p_n$ such that $|f(x) - p_n(x)| < 1/n$ for all $x$ such that $\|x\| \leq n$. The sequence $\{p_n\}$ converges to $f$ uniformly on bounded sets. This concludes the proof.

Remark. Weakly continuous on bounded sets means: the restriction of $f$ to any bounded subset of $X$ is relatively weakly continuous. If a function is weakly continuous, then it is weakly continuous on bounded sets. The converse is not true. Let $X = H$ be a Hilbert space, let $e_1, e_2, \ldots, e_n, \ldots$ be a basis of $H$ and let

$$f(x) = \sum_{n=1}^{\infty} \frac{(e_n(x))^n}{(2n)!}.$$ 

If this function were weakly continuous at the origin of $H$ then it would be bounded on some subspace $M$ of finite codimension in $H$. For $x \in M$ fixed,

$$g(z) = \sum_{n=1}^{\infty} \frac{(e_n(x))^n}{(2n)!} z^n$$

is an entire function of order $1/2$ in $z$. Such a function cannot be bounded for all real $z$ unless it is a constant, in this case zero. Hence $M = \{0\}$ which is a contradiction. I am grateful to the referee for pointing out this nice example and for other constructive comments.

We say that a Banach space $X$ has property (B) if there is a sequence $P_n: X \to X$ of bounded linear operators such that
(i) $P_n(X)$ is finite dimensional for each $n$.
(ii) $P_n^2 = P_n$.
(iii) $P_n(x) = x$ for every $x \in X$.
(iv) $P_n^*(u) = u$ for every $u \in X^*$.

Every Banach space with a biorthogonal basis has property (B). In particular, every Hilbert space has property (B). The theorem below has been proved by Vainberg in [1] for reflexive Banach spaces with a biorthogonal basis. Our proof is similar to his.

**Theorem 4.** Let $X$ be a reflexive Banach space with property (B). Let $T: X \to X^*$ be a continuous operator. Then $T$ is weakly continuous on bounded sets if and only if $T \circ P_n \to T$ uniformly on bounded sets.

**Proof.** By the uniform boundness principle there is a constant $M$ such that $\|P_n\| \leq M$ and $\|P_n^*\| \leq M$ for all $n$. Let us show that if $x_n \to x$ then $P_n x_n \to x$. Now, $\langle P_n x_n, u \rangle = \langle x_n, P_n^* u \rangle = \langle x_n, u \rangle + \langle x_n, P_n^* u - u \rangle$. Since $P_n^* u \to u$, and $\{x_n\}$ is bounded, it follows that $\langle P_n x_n, u \rangle \to 0$ for every $u \in X^*$.

Let us assume that $T$ is weakly continuous on bounded sets and suppose that $T \circ P_n$ does not converge to $T$ uniformly on bounded sets. Then, for some $\varepsilon > 0$ and for every integer $k$ there exist an integer $n_k$ and a point $x_k$ such that $\|x_k\| \leq M$ for some $M > 0$ and $\|T P_{n_k} x_k - T x_k\| \geq \varepsilon$, $k = 1, 2, \ldots$. Since $X$ is reflexive, there is a subsequence (still denoted by $\{x_k\}$) such that $x_k \to x_0$ for some $x_0$. This would imply that $P_{n_k} x_k \to x_0$ and $T P_{n_k} x_k \to T x_0$. Thus, we have reached a contradiction.

Let us assume that $T \circ P_n \to T$ uniformly on bounded sets. Since $P_n^* PB$ is weakly continuous by Theorem 2 and $\{x | \|x\| \leq M\}$ is weakly compact for every $M > 0$, it follows that $T$ is weakly continuous on bounded sets.

**3. Uniformly differentiable functions.** Let $X$ and $Y$ be Banach spaces and let $f: X \to Y$ be of class $C^1$. We say that $f$ is uniformly differentiable in a subset $A \subset X$ if for every $\varepsilon > 0$ there is some $\delta > 0$ such that

$$\|f(a + h) - f(a) - \langle f'(a), h \rangle\| \leq \varepsilon \|h\|$$

whenever $\|h\| \leq \delta$ for all $a \in A$.

**Theorem 5.** Let $X$ be a reflexive Banach space with property (B). Let $f: X \to \mathbb{R}$ be weakly continuous and uniformly differentiable on bounded sets. Let $T = f': X \to X^*$. Then $P_n^* \circ T \to T$ uniformly on bounded sets.

**Proof.** Let $I: X \to X$ be the identity operator and let $h = (I - P_n)(z)$. 

Let \( O(x, h) = f(x + h) - f(x) - \langle T(x), h \rangle \). Since \( \langle T(x), h \rangle = \langle (I - P_n^*) T x, h \rangle \) we get,

\[
\langle (I - P_n^*) T x, h \rangle \leq \left| f(x + h) - f(x) \right| + \left| O(x, h) \right|
\]

(1)

\[ \leq \left| f(x + h) - f(P_n (x + h)) \right| + \left| f(P_n (x + h)) - f(x) \right| + \left| O(x, h) \right|. \]

Let \( \varepsilon > 0 \) be given and let \( M > 0 \). Since \( f \) is uniformly differentiable on bounded sets there is some \( \delta > 0 \) such that

\[
\left| O(x, h) \right| < (\varepsilon/3) ||h|| \text{ if } ||h|| < \delta
\]

(2) for all \( x \) such that \( ||x|| \leq M \). Now, by the uniform boundness principle there is some \( N \) such that \( ||I - P_n|| \leq N \) for all \( n \). Since \( f \) is weakly continuous, there is some \( n_0 \) (see Theorem 4) such that

\[
\left| f(x + h) - f(P_n (x + h)) \right| < \varepsilon \delta / 3N
\]

(3) \( n \geq n_0 \)

for all \( x \) such that \( ||x|| \leq N \) and all \( h \) such that \( ||h|| < \delta \). Let \( y \in X \) such that \( ||y|| = 1 \) and let \( h = (\delta / N) (I - P_n)y \). then \( ||h|| \leq \delta \) and

\[
\langle (I - P_n^*) T x, y \rangle = \langle (I - P_n^*) T x, (I - P_n)y \rangle = \langle (I - P_n^*) T x, h \rangle ;
\]

(4) from (1), (2), (3) and (4) we get

\[
\left| (I - P_n^*) T x, y \right| < \varepsilon \text{ if } n \geq n_0
\]

(5) for all \( x \) such that \( ||x|| \leq M \). This concludes the proof.

**Theorem 6.** Let \( X \) be a reflexive Banach. Assume \( f: X \to \mathbb{R} \) is weakly continuous and uniformly differentiable on bounded sets. Then \( f' = T: X \to X^* \) is weakly continuous.

**Proof.** We have to show that \( x_n \to x \) implies \( T x_n \to T x \). Since \( \{ x_n \} \) is bounded, we can assume that everything takes place in some bounded set \( B = \{ x \mid ||x|| \leq M \} \). Let \( \varepsilon > 0 \). By uniform differentiability we can make

\[
\left| O(x, h) \right| < 4^{-1} \varepsilon ||h||
\]

(1) if \( ||h|| \leq \delta < 1 \) for some \( \delta \) and all \( x \in B \). Since \( f \) is weakly uniformly continuous in \( 2B \), there is a weak neighborhood \( V(0) \) such that,

\[
\left| f(y) - f(z) \right| < 4^{-1} \delta \varepsilon
\]

(2) for all \( y, z \in 2B \) such that \( y - z \in V(0) \). Now,
\langle Tx_n, h \rangle = f(x_n + h) - f(x_n) - O(x_n, h)
\langle Tx, h \rangle = f(x + h) - f(x) - O(x, h)

and

\| \langle Tx_n - Tx, h \rangle \| \leq | f(x_n + h) - f(x + h) | + | f(x_n) - f(x) |
+ | O(x_n, h) | + | O(x, h) |.

(3)

Let \( n_0 \) be such that \( x_n - x_0 \in V \) if \( n \geq n_0 \). From (1), (2) and (3) we obtain

\| \langle Tx_n - Tx, h \rangle \| < \epsilon \delta

if \( n \geq n_0 \) and \( \| h \| = \delta \). This shows that \( \langle Tx_n - Tx, h \rangle \| < \epsilon \) if \( n \geq n_0 \) and

\| T x_n - T x \| < \epsilon \) if \( n \geq n_0 \). This concludes the proof.

**Theorem 7.** Let \( X \) be a reflexive Banach with property (B). Let \( f: X \to \mathbb{R} \) be weakly continuous and uniformly differentiable on bounded sets. Then \( P_n^* T P_n \to T \) uniformly on bounded sets.

**Proof.** We have for any \( x \in X \)

\| T x - P_n^* T P_n(x) \| \leq \| T x - P_n^* T x \| + \| P_n^* T x - P_n^* T P_n x \|.

The first term in the right-hand side converges to zero uniformly on bounded sets (Theorem 5). The second term is lesser or equal than \( \| P_n^* T x - T P_n x \| \) which converges to zero uniformly on bounded sets by Theorems 6 and 4 and the fact that \( \| P_n \| \leq M \) for some constant \( M \) and all \( n \). This concludes the proof.

**4. The main theorem.**

**Theorem 8.** Let \( X \) be a reflexive Banach space with property (B). Let \( \mathcal{P}(X) \) be the algebra of polynomials in \( X \) with the topology of uniform convergence on bounded sets of a function and its derivative. Then the closure of \( \mathcal{P}(X) \) is the algebra of weakly continuous functions on bounded sets which are uniformly differentiable on bounded sets.

**Proof.** Let \( f \) be weakly continuous and uniformly differentiable on bounded sets, and let \( f_n = f \circ P_n \). Then \( f_n^* = P_n^* \circ f^* \circ P_n \) (Theorem 2), \( f_n \to f \) uniformly on bounded sets (Theorem 4) and \( f_n^* \to f^* \) uniformly on bounded sets (Theorem 7). Therefore, we can assume that that \( f = g \circ P \) where \( P \) is a projection into some finite-dimensional subspace and \( g \) is uniformly differentiable on bounded sets. Let \( P(X) = H \) and let \( h \) be the restriction of \( g \) to \( H \), that is, \( h(x) = g(x) \) for all \( x \in H \). Since \( H \) is finite dimensional there is a sequence \( \{ q_n \} \) of
polynomials in $H$ such that $q_n \to h$, $q'_n \to h'$ uniformly on bounded sets (this follows from a classical theorem of Bernstein). Let $p_n = q_n \circ P$. Then,

$$|f(x) - p_n(x)| = |gP(x) - q_nP(x)| \to 0$$

uniformly on bounded sets. Now, let $u \in X$ such that $\|u\| = 1$. Then,

$$|\langle f'(x) - p'_n(x), u \rangle| = |\langle g'P(x) - q'_nP(x), Pu \rangle|$$
$$= |\langle h'P(x) - q'_nP(x), Pu \rangle|$$
$$\leq \|P\| \|h'P(x) - q'_nP(x)\|.$$

Therefore, $\|f'(x) - P_n'(x)\| \leq \|P\| \|h'P(x) - q'_nP(x)\| \to 0$ uniformly on bounded sets. This concludes the proof.

**References**


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