ON BERMAN'S VERSION OF THE LÉVY-BAXTER THEOREM

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In this note we derive the numerical value of the constant $B_k$ in Berman's version of the Lévy-Baxter theorem [2]. Let $X(t)$, $t = (t_1, t_2, \cdots, t_k), -\infty < t_1, \cdots, t_k < \infty$, $||t|| = (t_1^2 + t_2^2 + \cdots + t_k^2)^{1/2}$, be Lévy's Brownian process of $k$ parameters: it is a Gaussian process with mean 0 and covariance function

\begin{equation}
\rho(s, t) = E\{X(s)X(t)\} = (1/2) \{||s|| + ||t|| - ||s - t||\}.
\end{equation}

For each integer $n \geq 1$, let the unit cube $\{t: 0 \leq t_1 \leq 1, \cdots, 0 \leq t_k \leq 1\}$ be broken up into $2^{nk}$ cubes whose edges have the common length $2^{-n}$ and whose corner-points are of the form $(i_12^{-n}, \cdots, i_k2^{-n})$, where the $i$'s are integers between 0 and $2^n$. Let $Y_{i,n}$ denote the $k$th-order difference of the sample function $X$ over the cube $C(i, n) = \{t: (i_1 - 1)2^{-n} \leq t_1 \leq i_12^{-n}, \cdots, (i_k - 1)2^{-n} \leq t_k \leq i_k2^{-n}\}$:

\begin{equation}
Y_{i,n} = \Delta_1 \cdots \Delta_k X = X(i_12^{-n}, \cdots, i_k2^{-n}) - \sum_{r=1}^{k} p_r + \sum_{r<s} p_{rs}
\end{equation}

where $p_{rs} \cdots t$ denotes $X(c_1, \cdots, c_k)$ for $c_r = (i_r - 1)2^{-n}, c_s = (i_s - 1)2^{-n}, \cdots, c_t = (i_t - 1)2^{-n}$ and the remaining $c_j$ equal $i_j2^{-n}$. S. Berman [2] proved: For $n \geq 1$, let $\sum |Y_{i,n}|^{2k}$ be the sum of the $2k$th powers of the $Y_{i,n}$ over all cubes $C(i, n)$. Its limit, for $n \to \infty$, exists with probability 1 and is equal to a numerical constant $B_k$.

The theorem below gives the numerical value of $B_k$.

**Theorem.**

\begin{equation}
B_k = \frac{(2k)!}{k!2^k} \left[ \frac{1}{2^k} \sum_{r=1}^{k} (-1)^{r-1} \binom{k}{r} \sqrt{r} \right]^k
\end{equation}

where

\begin{equation}
\binom{k}{r} = \frac{k!}{r!(k-r)!}.
\end{equation}

**Proof.** Berman showed in [2] that

\begin{equation}
B_k = ((2k)!/k!2^k) D_k^k
\end{equation}

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where \( D_k \) is the variance of the \( k \)th-order difference of \( X(\cdot) \) over the corner-points of the unit cube. We shall show that

\[
D_k = 2^{k-1} \sum_{r=1}^{k} (-1)^{r-1} \binom{k}{r} \sqrt{r}.
\]

Represent the \( 2^k \) different corner points by \( t_1, \ldots, t_{2^k} \). Let \( e_j, j = 1, \ldots, 2^k \) denote \( \pm 1 \) according to the rule: \( e_j = +1 \) if there are an even number of zeros in the \( k \) coordinates of \( t_j \), whereas, \( e_j = -1 \) if there are an odd number of zeros in the \( k \) coordinates of \( t_j \). We can then write \( D_k \) as

\[
D_k = \text{Var} \left[ \sum_{j=1}^{2^k} e_j X(t_j) \right] = \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} e_i e_j p(t_i, t_j)
\]

\[
= \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} e_i e_j \left( \| t_i \| + \| t_j \| - \| t_i - t_j \| \right)
\]

\[
= \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} e_i e_j \| t_i \| - \left( \frac{1}{2} \right) \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} e_i e_j \| t_i - t_j \|.
\]

There are \( C_{k,j} \) distinct ways of forming \( k \)-tuples consisting of exactly \( j \) zeros and \( k-j \) ones, thus

\[
\sum_{j=1}^{2^k} e_j = \sum_{j=0}^{k} \binom{k}{j} (-1)^j = 0
\]

(For the last equality see [3, p. 63].) We therefore have

\[
\sum_{i=1}^{2^k} \sum_{j=1}^{2^k} e_i e_j \| t_i \| = \sum_{j=1}^{2^k} e_j \sum_{i=1}^{2^k} e_i \| t_i \| = 0.
\]

Now, since the \( t_j \)'s are corner points of the unit cube in \( k \)-dimensional Euclidean space, we have \( \| t_i - t_j \| = \sqrt{r} \) where \( r \) is the number of coordinates in \( t_i \) that differ from the corresponding coordinates in \( t_j \). Also, note that if \( \| t_i - t_j \| = \sqrt{r} \), then \( e_i e_j = (-1)^r \). Thus,

\[
- \left( \frac{1}{2} \right) \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} e_i e_j \| t_i - t_j \| = - \left( \frac{1}{2} \right) \sum_{r=1}^{k} \binom{k}{r} 2^k \sqrt{r} (-1)^r
\]

\[
= 2^{k-1} \sum_{r=1}^{k} (-1)^{r-1} \binom{k}{r} \sqrt{r}.
\]

Equations (4), (6), (8) and (9) yield the desired result of the theorem.
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REFERENCES


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