ON BERMAN'S VERSION OF THE LÉVY-BAXTER THEOREM

PEGGY TANG STRAIT

In this note we derive the numerical value of the constant $B_k$ in Berman's version of the Lévy-Baxter theorem [2]. Let $X(t)$, $t = (t_1, t_2, \cdots, t_k), -\infty < t_1, \cdots, t_k < \infty, \|t\| = (t_1^2 + t_2^2 + \cdots + t_k^2)^{1/2}$, be Lévy's Brownian process of $k$ parameters: it is a Gaussian process with mean 0 and covariance function

$$\rho(s, t) = E\{X(s)X(t)\} = (1/2)\{\|s\| + \|t\| - \|s - t\|\}. \quad (1)$$

For each integer $n \geq 1$, let the unit cube $\{t: 0 \leq t_1 \leq 1, \cdots, 0 \leq t_k \leq 1\}$ be broken up into $2^{nk}$ cubes whose edges have the common length $2^{-n}$ and whose corner-points are of the form $(i_12^{-n}, \cdots, i_k2^{-n})$, where the $i$'s are integers between 0 and $2^n$. Let $Y_{i,n}$ denote the $k$th-order difference of the sample function $X$ over the cube $C(i, n) = \{t: (i_1-1)2^{-n} \leq t_1 \leq i_12^{-n}, \cdots, (i_k-1)2^{-n} \leq t_k \leq i_k2^{-n}\}$:

$$Y_{i,n} = \Delta_1 \cdots \Delta_k X = X(i_12^{-n}, \cdots, i_k2^{-n}) - \sum_{r=1}^{k} p_r + \sum_{r<s}^{k} p_{rs}$$

$$- \cdots + (-1)^k X((i_1-1)2^{-n}, \cdots, (i_k-1)2^{-n}) \quad (2)$$

where $p_{rs} \cdots t$ denotes $X(c_1, \cdots, c_k)$ for $c_r = (i_r-1)2^{-n}$, $c_s = (i_s-1)2^{-n}$, $\cdots$, $c_t = (i_t-1)2^{-n}$ and the remaining $c_j$ equal $i_j2^{-n}$. S. Berman [2] proved: For $n \geq 1$, let $\sum|Y_{i,n}|^{2k}$ be the sum of the $2k$th powers of the $Y_{i,n}$ over all cubes $C(i, n)$. Its limit, for $n \to \infty$, exists with probability 1 and is equal to a numerical constant $B_k$.

The theorem below gives the numerical value of $B_k$.

**Theorem.**

$$B_k = \frac{(2k)!}{k!2^k} \left[ 2^{k-1} \sum_{r=1}^{k} (-1)^{r-1} \binom{k}{r} \sqrt{r} \right]^k \quad (3)$$

where

$$\binom{k}{r} = \frac{k!}{r!(k-r)!}. \quad (4)$$

**Proof.** Berman showed in [2] that

$$B_k = ((2k)!/k!2^k)D_k^k \quad (4)$$

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where $D_k$ is the variance of the $k$th-order difference of $X(\cdot)$ over the corner-points of the unit cube. We shall show that

$$D_k = 2^{k-1} \sum_{r=1}^{k} (-1)^{r-1} \binom{k}{r} \sqrt{r}.$$ 

Represent the $2^k$ different corner points by $t_1, \ldots, t_{2^k}$. Let $e_j, j = 1, \ldots, 2^k$ denote $\pm 1$ according to the rule: $e_j = +1$ if there are an even number of zeros in the $k$ coordinates of $t_j$, whereas, $e_j = -1$ if there are an odd number of zeros in the $k$ coordinates of $t_j$. We can then write $D_k$ as

$$D_k = \text{Var} \left[ \sum_{j=1}^{2^k} e_j X(t_j) \right] = \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} e_ie_j \rho(t_i, t_j).$$

$$= \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} e_ie_j (1/2) \left( ||t_i|| + ||t_j|| - ||t_i - t_j|| \right)$$

$$= \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} e_ie_j ||t_i|| - \left(1/2\right) \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} e_ie_j ||t_i - t_j||.$$ 

There are $C_{k,j}$ distinct ways of forming $k$-tuples consisting of exactly $j$ zeros and $k-j$ ones, thus

$$\sum_{j=1}^{2^k} e_j = \sum_{j=0}^{k} \binom{k}{j} (-1)^{j} = 0$$

(For the last equality see [3, p. 63].) We therefore have

$$\sum_{i=1}^{2^k} \sum_{j=1}^{2^k} e_ie_j ||t_i|| = \sum_{j=1}^{2^k} e_j \sum_{i=1}^{2^k} e_i ||t_i|| = 0.$$ 

Now, since the $t_j$'s are corner points of the unit cube in $k$-dimensional Euclidean space, we have $||t_i - t_j|| = \sqrt{r}$ where $r$ is the number of coordinates in $t_i$ that differ from the corresponding coordinates in $t_j$. Also, note that if $||t_i - t_j|| = \sqrt{r}$, then $e_ie_j = (-1)^r$. Thus,

$$- \left(1/2\right) \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} e_ie_j ||t_i - t_j|| = - \left(1/2\right) \sum_{r=1}^{k} \binom{k}{r} 2^k \sqrt{r} (-1)^r$$

$$= 2^{k-1} \sum_{r=1}^{k} (-1)^{r-1} \binom{k}{r} \sqrt{r}.$$ 

Equations (4), (6), (8) and (9) yield the desired result of the theorem.
References


City University of New York, Queens College