THE BREADTH AND DIMENSION OF A TOPOLOGICAL LATTICE

TAE HO CHOE

E. Dyer and A. Shields [7] conjectured that if $L$ is a compact connected metrizable distributive topological lattice, then $\dim(L)$ is equal to the breadth of $L$. L. W. Anderson [1] has proved that if $L$ is a locally compact, (chain-wise) connected distributive topological lattice then the breadth of $L$ is less then or equal to the codimension of $L$.

In this note we shall show that if $L$ is a locally compact connected distributive topological lattice of inductive dimension $n$ and if the set of points at which $L$ has dimension $n$ has nonvoid interior, then the breadth of $L$ is also $n$.

Our terminology and notation used in this paper are the same as in [1] and [4].

It is well known that the number of elements in a finite Boolean algebra is always a power of two, and if there are $2^n$ elements then there are exactly $n$ atoms.

It is also known that the center $C$ of any lattice with 0 and 1 forms a Boolean lattice with the same 0 and 1, and that if $L$ is a compact connected topological lattice of codimension $n$ then the cardinal of $C$ of $L$, hereafter denoted by $\text{Card}(\text{Cen}(L))$, is at most $2^n$, [4].

The following two lemmas have appeared in [4].

**Lemma 1.** Let $L$ be a compact connected distributive topological lattice. Then $L$ is isomorphic (homeomorphic and lattice-isomorphic) with a Cartesian product of $n$ nondegenerate compact connected chains (hereafter called an $n$-cube) iff (i) the center of $L$ has exactly $n$ atoms and (ii) for each atom $x$ of the center $x/L$ is a chain.

**Lemma 2.** Let $L$ be a distributive topological lattice with 0 and 1. If \{\(x_1, \cdots, x_n\)\} is the set of all the atoms of the center of $L$ and if $x$ and $y$ are two incomparable elements in $x/L$ for some $i$, then $\text{Card}(\text{Cen}(L)) < \text{Card}(\text{Cen}(\{x \land y, \ x \lor y \lor c(x_i)\}))$, where $c(x_i)$ is the complemented element of $x_i$ in the center of $L$.

Hereafter, inductive dimension is referred to as dimension. In a space $L$ of dimension $n$, $L_n$ denotes the set of all points at which $L$ has dimension $n$. If $L$ is a topological lattice and $[x, y] \subseteq L$, and if $\text{Card}(\text{Cen}([x, y])) = 2^n$, then $\text{Ca}([x, y])$ will denote the cardinal $n$ of the atoms of $\text{Cen}([x, y])$. It is known [6] that the codimension is

Received by the editors June 17, 1968.
less than or equal to the inductive dimension in a locally compact Hausdorff space.

**Theorem.** If $L$ is a locally compact connected distributive topological lattice of dimension $n$ and if $L_n$ has a nonvoid interior, then $L$ contains a subset which is isomorphic with an $n$-cube.

**Proof.** Let $W_1$ be a nonvoid open subset of $L$ and $W_1 \subseteq L_n$. For an element $w \in W_1$ choose neighborhoods $V_i$ and $U_i$ of $w$ and a closed interval $[a, b]$ such that $V_i$ is convex, $U_i^*$ compact and $V_i \subseteq [a, b] \subseteq U_i^* \subseteq W_1$, (see [5]), where $U_i^*$ denotes the closure of $U_i$. Then $M = [a, b]$ is a compact connected distributive topological lattice of dimension $n$ in its relative topology. Consider $N = \{m | m = Ca([x, y])$ for a closed interval $[x, y] \subseteq V_1\}$. Then $N$ is bounded by $n$. Let $m$ be the greatest integer of $N$, $m = Ca([\alpha, \beta])$, and let $\{x_1, \ldots, x_m\}$ be the set of atoms of the center of $[\alpha, \beta]$. Now we shall show that for each $i$, $I_i = x_i \cap [\alpha, \beta]$ is a nondegenerate compact connected chain. In fact, $[\alpha, \beta]$ is a distributive sublattice. Suppose that $I_i$ is not a chain for some $i$. Then, by Lemma 2, there exists a closed interval $[\alpha', \beta']$ in $[\alpha, \beta]$ such that $m \leq Ca([\alpha', \beta'])$. This is a contradiction. For each $i$, $I_i$ is compact and connected because $[\alpha, \beta]$ is. Moreover, the relative topology of each $I_i$ is its intrinsic topology, (see [3]). Since $M$ is compact, $[\alpha, \beta]$ is a compact connected distributive topological lattice in its relative topology. By Lemma 1, it follows that $[\alpha, \beta]$ is isomorphic with an $m$-cube $I^m = I_1 \times I_2 \times \cdots \times I_m$ under a mapping $f$.

Suppose $m < n$. Let $p$ be an element of $[\alpha, \beta]$ such that the $i$th coordinate of $f(p)$ is an inner point of $I_i$, $i = 1, 2, \ldots, m$, (such a point $p$ is called an inner point of $I^m$). Now we choose a convex neighborhood $U_2$ of $p$ such that

$$U_2 \subseteq L \cup \{ (c \land L) \cup (c \lor L) | c \in Cen([\alpha, \beta]) \text{ and } c \neq \alpha, c \neq \beta \}.$$ Setting $U = U_2 \cap V_1$, we choose a neighborhood $W$ of $p$ such that $W \lor W \subseteq U$ and $W \land W \subseteq U$. If we set $V = U \cap [\alpha, \beta]$, then $V$ is a convex neighborhood of $p$ in $[\alpha, \beta]$, and the dimension of $V$ is at most $m$. Thus $W \land V \neq \emptyset$. Then $p \leq \gamma < z \lor p$. Since $U$ is convex and contains $p$ and $z \lor p$, we have $\gamma \in V = U \cap [\alpha, \beta]$. Clearly $\gamma$ is an inner point of $I^m$. So $[\gamma, \beta]$ is also isomorphic with a sub $m$-cube $J^m$ of $I^m$. Consider a compact connected chain $C(\gamma, \delta)$ from $\gamma$ to $\delta = z \lor p$. Then $C(\gamma, \delta)$ is nondegenerate. Moreover, we have $\delta \land [\gamma, \beta] = \{ \gamma \}$. Therefore, the mapping $g: Q^{m+1} = [\gamma, \beta] \times C(\gamma, \delta)$
$P^{m+1} = [\gamma, \beta] \vee C(\gamma, \delta)$ defined by $g(x, s) = x \vee s$ is an isomorphism, where $g^{-1}(y) = (y \wedge \beta, y \wedge \delta)$. Since $g^{-1}(U \cap P^{m+1})$ is nonvoid and open in $Q^{m+1}$, and $Q^{m+1}$ is isomorphic with an $(m+1)$-cube $J^m \times C(\gamma, \delta)$, $U \cap P^{m+1}$ contains a closed interval which is isomorphic with an $(m+1)$-cube, and which is contained in $V_1$. Moreover, $Ca(J^m \times C(\gamma, \delta)) = m + 1$. This is a contradiction to the fact of the maximality of $m$. For the case that $\varepsilon \wedge \rho \notin V$, the dual argument also leads to a contradiction. Hence we have $m = n$ which completes the proof.

**Corollary 1.** Under the same hypothesis of the Theorem, the breadth, the dimension and the codimension are all the same.

It is known [8] that if $L$ is a compact separable space of dimension $n$, then $L_n$ has dimension $n$, and that any $n$-dimensional subset of Euclidean $n$-space has nonvoid interior.

Hence we have the following corollary:

**Corollary 2.** If $L$ is a locally compact connected distributive topological lattice of dimension $n$ which is topologically contained in Euclidean $n$-space, then $L$ contains a subset which is isomorphic with an $n$-cell.

**Remark.** It is a natural question that if $L$ is a compact connected distributive topological lattice of dimension $n$, then $L_n$ has nonvoid interior. This question is still open. Professor J. T. Borrego has, however, given an example of a compact connected, locally connected, acyclic topological semilattice of 2-dimensions whose $L_2$ has void interior.

**References**