RECURSIVE FUNCTIONS DEFINED BY
ORDINAL RECURSIONS

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A function, \(\phi(x_1, \ldots, x_n, y)\), will be said to be defined by (un-
nested) \(<\)-ordinal recursion (o.r. \(<\)), with respect to the irreflexive
well-ordering \(<\) of the set of natural numbers if it is defined by a
sequence of Kleene's five schemes for primitive-recursive (p.r.) func-
tions \([\text{KIM}, \text{p. 219}]\) and the ordinal recursion scheme,

\[
\begin{align*}
  f(\bar{a}, 0) &= g(\bar{a}) \\
  f(\bar{a}, b') &= k(\bar{a}, b, f(\bar{a}, h(\phi(\bar{a}, b))))
\end{align*}
\]

where \(k, g\) and \(l\) denote o.r. \(<\) functions and we assume that 0 is the
least element of \(<\), and \(h_\prec\) denotes the function \(\theta_\prec(b, c) = c\) if \(c_\prec b\),
0 otherwise.

If the well-ordering \(<\) is general recursive, by Church's Thesis, the
o.r. functions are general recursive. If \(<\) is the natural order, the
o.r. \(<\) functions are the p.r. functions, and by making \(<\) a "natural"
p.r. well-ordering of type \(\omega\), we obtain a function which is not primi-
tive recursive, as is shown, e.g., in Péter's paper, \([\text{P 50, p. 271}]\).
By increasing the order type of \(<\), it is possible to construct an expand-
ing hierarchy of recursive functions where computation difficulty is
related to order-type. Hopes of a natural connection between order-
type and computation difficulty were destroyed by results of Rout-
ledge \([\text{R 53}]\) and Myhill \([\text{M 53}]\), which showed that each general
recursive function is definable by o.r. \(<\) for a general recursive well-
ordering, \(<\), of type \(\omega\) (in Myhill's result, \(<\) is p.r.). The proof of
Myhill's result appears in Liu \([\text{L 60}]\).

For a well-ordering (w.o.) of type \(\omega\), we use the symbols \(\text{pd}_\prec\) and
\(\text{sc}_\prec\) to denote the predecessor and successor functions, respectively,
on \(<\). An examination of Liu's w.o., \(<\), used in the definition by
o.r. \(<\) of the general recursive function, (g.r.), \(\phi\), (in Liu's notation)
shows that:

\[
\phi(x') = U((\text{sc}_\prec(2^x + \sigma(x, 0))))_1)
\]

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RECURSIVE FUNCTIONS

where \( \sigma \) is a p.r. function. Hence, while Liu's \( \text{pd} < \) is p.r., \( \phi \) is primitive recursive in \( \text{sc} < \), and thus \( \text{sc} < \) cannot generally be p.r. This is the basis of a comment at the end of the paper [L 62], attributed to Hartley Rogers, Jr., that the recursive complexity of \( \phi \) is "absorbed" by \( \text{sc} < \). In this paper, we show that it need not be true that \( \text{sc} < \) "absorbs" the complexity of \( \phi \) by duplicating the Myhill-Liu result for a p.r.w.o. of type \( \omega \), for which both \( \text{pd} < \) and \( \text{sc} < \) are also p.r. This additional structure further makes it possible to duplicate Liu's "generalized concept" of primitive recursion [L 62] under conditions where the permutation, \( \text{pm} < \), and the successor function, \( \text{sc} < \), used there, are both primitive recursive. In addition, since our constructions are very simple, we shall not always produce p.r. defining equations for functions and predicates asserted to be p.r. We use Greek letters and mnemonic Latin combinations like \( \text{pd} \), \( \text{sc} \), for informally defined functions. Script Latin letters are used as function letters in recursion schemes only.

**Theorem 1.** For each general-recursive function \( \phi \), (temporarily of one variable) there exists a p.r.w.o., \( < \), of type \( \omega \), with both \( \text{pd} < \) and \( \text{sc} < \) p.r., and so that \( \phi \) is o.r. \( < \).

**Proof.** Let \( g \) be a fixed Gödel number for \( \phi \), and let \( \delta(n) = 2n + 1 \). Any 1-1 increasing p.r. function could be used in place of \( \gamma \). We choose \( \tau(x, y), \pi_1(z), \pi_2(z) \) as names for a triple of primitive recursive pairing and projection functions, \( (\forall z)(\tau(\pi_1(z), \pi_2(z)) = z) \), subject to the restrictions \( (\forall z)(z \geq \pi_2(z)) \) and \( \tau(0, 0) = 0 \). Further, we denote \( \tau(x, y) \) by \( \langle x, y \rangle \). Clearly, if we can construct \( < \) so that the following condition, \( C \), is satisfied,

\[
C: \text{ For each } n: (\mu z T_1(g, n, z), \gamma(n)) < (\mu z T_1(g, n, z) + 1, \gamma(n)) < \cdots < (1, \gamma(n)) < (0, \gamma(n))
\]

then, just as Liu does, we can use the o.r. \( < \) scheme.

\[
f(0) = 0 \\
f(b') = f(h < (b', \langle p_1(b') + 1, p_2(b') \rangle)) \text{ if } \\
\mu z T_1(g, \mu w w z (g(w) = p_2(b')), z) > p_1(b'), \\
= U(p_1(b')) \text{ if } \\
\mu z T_1(g, \mu w w z (g(w) = p_2(b')), z) = p_1(b'), \\
= 0 \text{ otherwise.}
\]

(NB. The defining clauses in this definition by cases are p.r. and the letters \( p \) and \( d \) denote the functions \( \pi \) and \( \delta \) respectively.) The func-
tion $\Psi$ defined by this scheme will be o.r. $\prec$, by condition C, and
$$(\forall x)(\phi(x) = \Psi(\langle 0, \gamma(x) \rangle)).$$ Hence, $\phi$ is o.r. $\prec$.

It remains to show that there does indeed exist a p.r.w.o., $\prec$, of
type $\omega$, satisfying condition C. We base our description upon the
following fundamental interval for $n$, $I_n$.

$$
\begin{aligned}
(0, 2n) & \prec (1, 2n) \prec \cdots \prec \langle m \mu T_1(g, n, z), 2n \rangle \\
I_n & \prec \langle m \mu T_1(g, n, z), 2n + 1 \rangle \prec \langle m \mu T_1(g, n, z) - 1, 2n + 1 \rangle \\
& \prec \cdots \prec (1, 2n + 1) \prec \langle 0, 2n + 1 \rangle.
\end{aligned}
$$

Let $I = \bigcup_{n=0}^{\infty} I_n$. We construct the order $\prec$ as follows:

$$I_0 \prec (S_0) \prec I_1 \prec (S_1) \prec \cdots \prec (S_{n-1}) \prec I_n \prec (S_n) \prec \cdots$$

where the “space” $(S_n)$ is filled by the number $n$ if $n \in I$, and left
vacant otherwise. It is easily seen that the predicate $z \in I$ is primitive
recursive and, hence, the ordering $\prec$ and the functions $\mu \mu \prec$, $\mu \mu \prec$
are all primitive recursive.

The restriction in Theorem 1 to functions of a single variable is
immediately removable in the usual manner, i.e. given $\phi(x_1, \cdots, x_n)$,
define $\phi'(x) = \phi((x)_1, \cdots, (x)_n)$; apply Theorem 1 to $\phi'$ and immedi-
ately, $\phi$ is o.r. $\prec$. With Liu, we let $\mu \mu \prec$ denote the permutation of
the natural numbers which maps the natural order into the order $\prec$,
for a w.o. of type $\omega$. The permutation $\mu \mu \prec$ has the property:

$$((\forall x)(\forall y)(x < y \iff \mu \mu \prec(x) < \mu \mu \prec(y)).$$

Since $\mu \mu \prec(0) = (0) = \mu \mu \prec(x') = \mu \mu \prec(x') = (0)$, we see that $\mu \mu \prec$
is primitive recursive in $\mu \mu \prec$. Hence, we have the following corollary.

**Corollary 2.** The permutation, $\mu \mu \prec$, associated with the well-
ordering of Theorem 1, is primitive recursive.

Thus Liu's "generalized primitive recursion" can be accomplished
with $\mu \mu \prec$ and $\mu \mu \prec$ both primitive recursive. If both $\mu \mu \prec$ and its
inverse, $(\mu \mu \prec)^{-1}$, are primitive recursive, then, clearly, $\phi$ is also
primitive recursive. More generally, we have the following theorem,
which is quite easy to prove, and which shows that the recursive com-
plexity of $\phi$ is now "absorbed" by $(\mu \mu \prec)^{-1}$, although this function
makes no explicit entry into Liu's "generalized" primitive recursion
for $\phi$.

**Theorem 3.** For any well-ordering of type $\omega$, $\prec$, and function $\phi$
which is o.r. $\prec$, $\phi$ is p.r. in $(\mu \mu \prec)^{-1}$.

**Proof.** We use induction on the length of the o.r. $\prec$ description
of $\phi$. Only the induction step is nontrivial and this only in case the o.r. $<$ ends in an application of the scheme
\[
f(\hat{a}, 0) = g(\hat{a}) \quad f(\hat{a}, b') = k(\hat{a}, b, f, (\hat{a}, h < (b', l(\hat{a}, b))))\]
where, the functions $\gamma, \kappa, \lambda$, represented by $g, k, l$ are o.r. $<$ and, by induction, p.r. in $(pm <)^{-1}$. By Theorem 6 of Routledge [R 53], $\phi$ is p.r. in $\gamma, \kappa, \lambda$ and in
\[
\Psi(y) = \mu z(\theta(z)(y, \lambda(x, y)) = 0)
\]
where $\theta(z) <$ represents the $z$th element of the splinter of $\theta <$ ($z$th iteration of the function), beginning with $y$. Since $\theta <$ is clearly p.r. in $(pm <)^{-1}$, and $\Psi(y) \leq (pm <)^{-1}(y)$, we see that $\phi$ is p.r. in $(pm <)^{-1}$.

In his review of [L 62], Myhill [M 66], asks whether there is a primitive recursive well-ordering of type $\omega$, $<$, for which $sc <$ is not even general recursive. Such orderings are easy to construct and, in fact, Markwald, [Ma 54], has shown how to construct such orders with neither $pd <$ nor $sc <$ general recursive. Augmenting Markwald's result, we can prove the following theorem. Note that if either $pd <$ or $sc <$ is general recursive, then $<$ is also general recursive, since $<$ is of type $\omega$.

**Theorem 4.** For each of (i), (ii), (iii), below, it is possible to find a general recursive well-ordering, $<$, of the natural numbers having type $\omega$, for which one and only one of (i), (ii), (iii), is primitive recursive.

(i) the order predicate $<$,
(ii) $pd <$,
(iii) $sc <$.

**Proof.** We prove only the case $sc <$ p.r., $<$, and $pd <$ not p.r. The symmetric case has a symmetric proof and the third, with $<$ p.r., is Markwald's result. We shall assume that the characteristic function of the order predicate $<$ is a function $\chi <$ with the property
\[
x < y \iff \chi(x, y) = 0.
\]
We construct a g.r.w.o., $<$ of type $\omega$, for which $sc <$ is p.r. by diagonalizing over the p.r. functions of two variables. To this end, let $\eta(n)$ be a p.r. function whose range is a set of Gödel numbers for two-placed primitive recursive functions and contains at least one Gödel number for each such function. For each $n$, we define two intervals
\[
A_n: \langle 0, n \rangle < \langle 2, n \rangle < \cdots < \langle 2m_n, n \rangle
\]
\[
B_n: \langle 1, n \rangle < \langle 3, n \rangle < \cdots < \langle 2m_n - 1, n \rangle
\]
where:

\[ m_n = 2^n 3^n + 1 \quad \text{and} \quad \mu_n = \mu z T \gamma (\eta(n), \langle 0, n \rangle, \langle 1, n \rangle, z). \]

Let \( C = \bigcup_{n=0}^\infty A_n \cup \bigcup_{n=0}^\infty B_n \) and let \( I_n \) be the interval \( A_n \triangleleft (S_n) \triangleleft B_n \) if \( U(m_n)_n \neq 0 \) and \( B_n \triangleleft (S_n) \triangleleft A_n \) if \( U(m_n)_n = 0 \). Again, the space \( (S_n) \) is filled by the number \( n \) if \( n \in C \) and is left blank otherwise. The predicate \( n \in C \) is primitive recursive and this, together with the fact that the decision whether \( A_{n+1} \triangleleft B_{n+1} \) or \( B_{n+1} \triangleleft A_{n+1} \) can be made primitive recursively in the last member of \( I_n \) shows that the successor function \( sc \) is primitive recursive.

The diagonal construction shows that the order predicate, \( \prec \), is not primitive recursive, for define

\[
\Psi(x, y) = \begin{cases} 0 & \text{if } \langle 0, x \rangle \prec \langle 1, y \rangle \\ 1 & \text{otherwise} \end{cases}
\]

if \( \prec \) were p.r., then \( \Psi(x, y) \) would be a p.r. function of two variables and we could choose \( n \) so that \( \eta(n) \) is a Gödel number of \( \Psi \). Then

\[
\{\eta(n)\} \langle \langle 0, n \rangle, \langle 1, n \rangle \rangle = \Psi(n, n) = 0
\]

\[
\Leftrightarrow U(m_n)_n = \{\eta(n)\} \langle \langle 0, n \rangle, \langle 1, n \rangle \rangle \neq 0.
\]

This contradiction shows that \( \prec \) is not p.r., and \( \Psi(x, x) \) is not p.r. Finally, since

\[
\Psi(x, x) = 0 \Leftrightarrow \text{pd } \prec \langle \langle 1, x \rangle \rangle = x \lor p(z) (\text{pd } \prec \langle \langle 1, x \rangle \rangle) = x
\]

then \( \text{pd } \prec \) cannot be p.r. without making \( \Psi(x, y) \) primitive recursive, and hence, \( \text{pd } \prec \) is not p.r.

References


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