A GRONWALL INEQUALITY FOR LINEAR STIELTJES INTEGRALS

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This paper provides a Gronwall type inequality which includes the one found by Schmaedeke and Sell [4].

Suppose that $S$ is an interval of real numbers containing zero and $OB$ is the collection of functions from $S$ to the real numbers each member of which is of bounded variation on each finite interval of $S$. The numeral 1 will also denote the constant function from $S$ which has only the value 1; if $x$ is in $S$, then $1_x$ denotes the function from $S$ which has the value 1 at $x$ and the value 0 elsewhere; and $0_x$ denotes the function $1 - 1_x$. Let $J$ be a function from $OB$ to the collection of functions from $S \times S$ to the real numbers having the following properties: if each of $f$ and $g$ is in $OB$ and \{x, y, z\} is in $S \times S \times S$ then

1. $J[f](x, y) + J[g](x, y) = J[f+g](x, y),$
2. if $r$ is a number then $J[r \cdot f](x, y) = r \cdot J[f](x, y),$
3. $J[f](x, y) + J[g](y, z) = J[f](x, z)$ provided that $x \leq y \leq z$ or $x \leq y \leq z$,
4. $J[f](x, z) \geq 0$ provided that $f(y) \geq 0$ for $x \leq y \leq z$ or $x \leq y \leq z$, and
5. if $x$ is in $S$ and $x \geq 0$ then each of $J[0_x](x, x^+) \text{ and } J[1_x](x^-, x)$ is less than 1; whereas, if $x$ is in $S$ and $x \leq 0$ then each of $J[1_x](x^+, x)$ and $J[0_x](x, x^-)$ is less than 1.

Theorem. If $J$ satisfies properties (1)-(5), there is a function $m$ from $S \times S$ to the real numbers having the following properties:

1. $m(x, y) \geq 1$ for each \{x, y\} in $S \times S$,
2. $m(x, y) \cdot m(y, z) = m(x, z)$ provided that $x \leq y \leq z$ or $x \geq y \geq z$,
3. $m(0, x) = 1 + J[m(0, \cdot)](0, x)$ for each $x$ in $S$, and
4. if $f$ is in $OB$, $P$ is a number, and $f(x) \leq P + J[f](0, x)$ for each $x$ in $S$, then $f(x) \leq P \cdot m(0, x)$ for each $x$ in $S$.

Remark. It is the purpose of this remark to show a connection between the above theorem and one of Schmaedeke and Sell. In [4], they investigate an inequality similar to that in part (iv) but use the mean Stieltjes integral and the Dushnik or interior integral (see also [3]). One-term approximating sums for these are indicated:

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\[ (\text{M}) \int_x^y f dg = \frac{f(x) + f(y)}{2} \cdot [g(y) - g(x)] \]

and

\[ (I) \int_x^y f dg = f(z) \cdot [g(y) - g(x)] \quad \text{where} \quad x < z < y \quad \text{or} \quad x > z > y. \]

If no member of \( S \) is negative, \( g \) is increasing and right continuous, and, for \( x \leq y \), \( J[f](x, y) \) is defined to be \( (\text{M}) \int_x^y f dg \), with \( J[f](y, x) = J[f](x, y) \), then \( J \) satisfies properties (1)–(4) and, also, property (5) in case \( g(z) - g(z^-) < 2 \) for all \( z \) different from zero. If \( P \) is a number, \( f \) is in \( OB \), and \( f(x) \leq P + (\text{M}) \int_x^y f dg \) for each \( x \) in \( S \), then \( f(x) \leq P + J[f](0, x) \) for each \( x \) in \( S \), since no member of \( S \) is negative. This inequality includes the inequality of \( [4, \text{p. 1219}] \). If, instead, \( J \) is defined in terms of the interior integral then properties (1)–(4) are, again, satisfied by \( J \) and property (5) makes no additional requirement due to the condition that \( g \) is right continuous. (See Remark 1 of \([4]\).)

Remark. With properties (1)–(3), a more familiar property which is equivalent to the conceptually simpler property (4) is

\[ (4\prime) \text{ if } f \text{ is in } OB \text{ and } \{x, z\} \text{ is in } S \times S \text{ and } m \text{ is a number such that } |f(y)| \leq m \text{ for all } y \text{ in } S \text{ such that } x \leq y \leq z \text{ or } x \geq y \geq z \text{ then } |J[f](x, z)| \leq m J[1](x, z) \text{ (compare } [2, \text{Axiom II}]). \]

To see that (4) implies \((4\prime)\), notice that each of \( m + f(y) \) and \( m - f(y) \) is nonnegative for \( x \leq y \leq z \) or \( x \geq y \geq z \); to see that \((4\prime)\) implies (4), notice that each of \( J[1_x] \) and \( J[0_x] \) has only nonnegative values and use the formulas in Theorem 1 and equation (24) of \([2]\). We shall use the fact that if \( f \) is in \( OB \) and \( \{x, y\} \) is in \( S \times S \) then \( |J[f](x, y)| \leq J[|f|](x, y) \) which follows from properties (1)–(4).

Indication of Proof of Theorem. The proof of parts (i), (ii), and (iii) of the theorem is only a slight modification of the ideas developed by MacNerney in \([2, \text{Theorems 1 and 2}] \) and used by the author in \([1, \text{Theorem 1.1}] \). For part (iv), suppose that \( f \) is in \( OB \), \( P \) is a number, and \( f(x) \leq P + J[f](0, x) \) for each \( x \) in \( S \). Define a sequence \( h \) with values in \( OB \) as follows: \( h_0 = f \) and, if \( n \) is a positive integer, then \( h_n(x) = P + J[h_{n-1}](0, x) \) for each \( x \) in \( S \). Let \( r \) be a function so that if \( x \) is in \( S \) then \( r(x) = \int_0^x d|h_2 - h_1| \). Let \( L \) be a sequence so that if \( x \) is in \( S \) then \( L_1(x) = r(x) \) and if \( n \) is a positive integer then \( L_{n+1}(x) = J[L_n](0, x) \). If \( n \) is a positive integer and \( a \) is in \( S \), then

\[ 0 \leq \sum_{p=1}^n L_p(a) \leq \sum_{p=1}^{n+1} L_p(a) \leq r(a) \cdot m(0, a). \]
Moreover, if $x$ is in $S$ and between 0 and $a$ and $n$ is a positive integer then $L_n(x) \leq L_n(a)$. Finally, if $n$ is a positive integer and $x$ is in $S$ then $|h_{n+1}(x) - h_n(x)| \leq L_n(x)$. Thus the sequence $h$ converges absolutely and, if $a$ is in $S$, uniformly on the set of all numbers in $S$ between 0 and $a$. Moreover, if $\lim h = U$ and $a$ is in $S$, then $U(a) = P + J[U](0, a)$; and $U(x) = Pm(0, x)$ for each $x$ in $S$. (To see this latter, recall [2, Theorems 2 and E].) We have, inductively, that if $p$ is a positive integer and $x$ is in $S$ then $f(x) \leq h_p(x)$. Consequently, $f(x) \leq Pm(0, x)$.

Remark. Using [1, Lemma 1.1] and similar techniques to the ones indicated above, we may obtain a more general inequality for a function $f$ which satisfies $f(x) \leq P + J[f](0, x) + g(x)$ where $g$ is in $OB$ and $g(0) = 0$.

Bibliography


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