ON THE CONJUGACY PROBLEM AND GREENDLINGER'S EIGHTH-GROUPS

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1. **Introduction.** In 1912 Max Dehn [**1**], [**2**] solved the conjugacy problem for the fundamental group G_k of a closed 2-manifold of genus k; this group can be finitely presented as follows:

$$G_k = gp(a_1, b_1, \cdots, a_k, b_k; a_1^{-1}b_1^{-1}a_1b_1 \cdots a_k^{-1}b_k^{-1}a_kb_k = 1).$$

In 1960 Greendlinger [**3**] solved the conjugacy problem for the class of eighth-groups; we note that this class contains the above groups G_k for $k \geqq 3$. Further generalizations of Dehn's work on the conjugacy problem were obtained again by Greendlinger [**4**] and also by Schupp [**11**] using methods developed by R. C. Lyndon [**9**].

In this paper, using a theorem of Solitar [**10,** Theorem 4.6], we solve the conjugacy problem for certain generalized free products of eighth-groups; namely,

MAIN THEOREM. *Let G be the free product of eighth-groups A_λ with a cyclic group H amalgamated where H is generated by a basic element h in each factor. Then G has a solvable conjugacy problem.*

We remark that the relevant notation and definitions appear either in §2 or in Magnus, Karrass and Solitar [**10**]. Our main technical result Lemma 3 uses a result of Greendlinger [**3,** Theorem V] which we restate below.

2. **Notation and definitions.** Let A be a group generated by $a_1, \cdots, a_\mu$ with defining relations $R_1 = 1, \cdots, R_\gamma = 1$. We denote this group by

$$A = gp(a_1, \cdots, a_\mu; R_1 = 1, \cdots, R_\gamma = 1),$$

where we call the R_i *relators*. We will assume (without loss in generality) that the relators R_i have been *symmetrized*, that is, that the R_i are cyclically reduced and include inverses and cyclic transforms of each other. We call A an *eighth-group* if, for any distinct relators $R_1 \equiv XY$ and $R_2 \equiv XZ$, the length of the common initial segment X is less than 1/8 of the length of either relator.

We use the following notation for words u and v in the group A:

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$l(u)$ for the length of u,
$u = v$ means u and v are the same element of the group A,
$u \approx v$ means u and v are freely equal,
$u \equiv v$ means u is identical to the word v,
$u \wedge v$ means u does not react with v, i.e. $w \equiv uv$ is freely reduced,
$u \subset v$ means u is a subword of v, i.e. $v \equiv w_1 u w_2$.

We say v *cyclically contains* u if a cyclic transform of v contains u as a subword.

We introduce the notation

$$u < p/q\ R$$

which means that u does not contain p/q or more of any defining relator R_i, that is, if u and R_i contain the same subword S then $l(S) < p/q l(R_i)$. We define $u \leqq p/q\ R$ analogously. We assume obvious properties of this definition such as: (i) if $u < 1/8\ R$ and $v < 1/8\ R$ then $uv < 2/8\ R$, (ii) if $u \subset v$ and $v < 1/8\ R$ then $u < 1/8\ R$, (iii) if $T < 1/8\ R$ and $T \subset R_i$ then $l(T) < 1/8\ l(R_i)$. We also assume, unless otherwise stated or implied, that all elements are written as freely reduced words $w \leqq 4/8\ R$.

DEFINITION. An element h in an eighth-group A is said to be *basic* if h is cyclically reduced, has infinite order, and is $< 1/8\ R$.

The condition $h < 1/8\ R$ implies, by Lemma 2 below, that $h^m < 2/8\ R$ for any integer m. (We note [**3**, Theorem VIII] that u in A has finite order only if there is a relator $R' \equiv S^n$ where u is a conjugate of a power of S.)

3. Theorems. Let u and v be cyclically reduced elements in the group G of the Main Theorem, and suppose u and v are conjugate in G. By Solitar's theorem [**10**, Theorem 4.6] u and v have the same length n. The following theorem gives additional conditions on u and v; here r denotes the length of the largest defining relator of those factors A_λ to which u or v belong.

THEOREM 1. *If $n = 1$, say $u \in A_i$ and $v \in A_j$, then either u and v are conjugate in $A_i = A_j$ or else u and v are conjugate to h^t in A_i and A_j respectively where*

$$|t| \leqq r^2 + rl(u).$$

If $n > 1$, then $h^m u^* h^{-m} = v^*$ where u^* and v^* are cyclic transforms of u and v respectively and

$$|m| \leqq l(u) + l(v) + 2 + r.$$

PROOF. Suppose $n=1$ and u and v are not conjugate in a factor. By Solitar's theorem there is a sequence of elements $u, h_1, h_2, \cdots, h_s, v$ where the h_i belong to the amalgamated subgroup H and consecutive terms of the sequence are conjugate in a factor. By Theorem 2 of [5], powers of h are in different conjugate classes in each factor; hence $h_1= \cdots =h_s=h^t$. Thus u and v are conjugate to h^t in A_i and A_j respectively. Furthermore, by the Theorem of [7], we have $|t| \leqq r^2+rl(u)$, as required.

Now suppose $n>1$. By Solitar's theorem, there exist cyclic transforms u^* and v^* of u and v respectively which are conjugate in G by an element in the amalgamated subgroup H, i.e. $h^m u^* h^{-m}=v^*$ for some integer m. It remains to show that m satisfies the condition stated in the theorem. Suppose u^* and v^* have normal forms $u^*=x_1x_2 \cdots x_n$ and $v^*=y_1y_2 \cdots y_n$. Then

$$h^m x_1 x_2 \cdots x_n h^{-m} = y_1 y_2 \cdots y_n.$$

If h commutes with each of the x's then we can choose $m=0$. On the other hand, let k be the smallest integer such that h does not commute with x_k. Then $y_k^{-1}h^m x_k$ belongs to the amalgamated subgroup H; say $y_k^{-1}h^m x_k=h^{-s}$. Then $h^m x_k h^s=y_k$. By Lemma 4 below,

$$|m| \leqq l(x_k) + l(y_k) + 2 + r \leqq l(u) + l(v) + 2 + r.$$

Thus Theorem 1 is proved.

PROOF OF MAIN THEOREM. By the above theorem, the conjugacy problem in G is reduced to a finite number of conjugacy problems in factors of G or to a finite number of word problems in G. The Main Theorem now follows from the fact that the conjugacy problem has been solved for the factors by Greendlinger [3] and the word problem has been solved for G by Lipschutz [6].

4. Lemmas. We first state Greendlinger's basic result on eighth-groups [3, Theorem V] which we need for our main technical result Lemma 3.

GREENDLINGER'S LEMMA. *Let W be a cyclically reduced nonempty word in an eighth-group A and suppose $W=1$. Then either W is a defining relator or else W cyclically contains disjointly*

(i) $>7/8$ *each of two R's,*
(ii) $>6/8$ *each of three R's,*
(iii) $>6/8$ *each of two R's and* $>5/8$ *each of two R's, or*
(iv) $>5/8$ *each of five R's.*

LEMMA 1. *Suppose in an eighth-group A there is a defining relator*

$R_1 \equiv W^n YT$ where $n>1$ and Y is an initial segment of W, i.e. $W \equiv YX$. Then either W has finite order, or else $W^{n-1}Y < 1/8\ R$ and $W^n Y < 2/8\ R$.

PROOF. Consider the defining relator $R_2 \equiv W^{n-1}YTW$. Note $W^{n-1}Y$ is a common initial segment of R_1 and R_2. Suppose $R_1 \not\equiv R_2$. Then by definition of an eighth-group $W^{n-1}Y < 1/8R$. Now $W \subset W^{n-1}Y$ because $n>1$. Hence $W<1/8\ R$ and therefore $W^nY<2/8\ R$. On the other hand, suppose $R_1 \equiv R_2$. Then W and $W^{n-1}YT$ commute as words in a free group and hence are each freely equal to powers of the same word U. Accordingly, R_1 is a power of U and hence U has finite order in the eighth-group A. But W is a power of U; hence W also has finite order in A.

LEMMA 2. *Let u be a cyclically reduced element of infinite order in an eighth-group A. If $u<1/8\ R$, then $u^m<2/8\ R$ for any integer m.*

PROOF. Suppose S is a subword of u^m and some defining relator R. Then $S \equiv (YX)^nZ$ where $u \equiv XY$ and $YX \equiv ZW$. We need to show that $l(S)<2/8\ l(R)$ or equivalently $S<2/8\ R$.

Case I. $n=0$; that is, $S \equiv Z$. Now Y, $X \subset u$ and $u<1/8\ R$; hence $S \equiv Z \subset YX$ and so $S<2/8\ R$.

Case II. $n>1$. By Lemma 1, $S<2/8\ R$.

Case III. $n=1$; that is, $S \equiv YXZ$. Suppose Z is an initial segment of Y. Then $XZ \subset XY \equiv u$; hence $XZ<1/8\ R$, and $Y<1/8\ R$. Accordingly, $S \equiv YXZ<2/8\ R$. On the other hand, suppose Y is an initial segment of Z; say $Z \equiv YU$. Then $S \equiv YXZ \equiv YXYU$ where U is a subword of X. We have

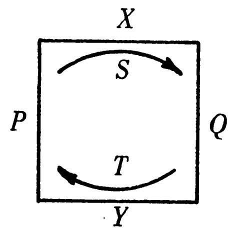
$$l(u) = l(XY) < 1/8\, l(R) \quad \text{and} \quad l(Y) + l(U) \leqq l(XY) < 1/8\, l(R).$$

Therefore $l(S)<2/8\ l(R)$.

Thus the lemma is proved.

LEMMA 3. *Let $W \equiv PXQY$ be a cyclically freely reduced nonempty word in an eighth-group A with P, $Q<2/8\ R$ and X, $Y \leqq 4/8\ R$. If $W=1$, then X and Y are nonempty and $l(P) \leqq \max l(R_i)$.*

PROOF. We write W around a square as in Figure 1. Since P, Q $<2/8\ R$, if W cyclically contains $>6/8$ of a defining relator then it must cross at least two corners of the square, and if W cyclically contains $>5/8$ of a defining relator then it must cross at least one corner of the square. Accordingly, W cannot satisfy conditions (ii), (iii) or (iv) of Greendlinger's lemma. Furthermore, if X is empty then: (1) $W \equiv PQY<8/8\ R$ and cannot be a defining relator, and (2) W cannot satisfy condition (i) of Greendlinger's lemma. Accordingly, X and similarly Y are nonempty words.



It remains to show that $l(P) \leqq \max l(R_i)$. If W is a defining relator then obviously $l(P) \leqq \max l(R_i)$. On the other hand, suppose W satisfies condition (i) of Greendlinger's lemma. Specifically, suppose W cyclically contains disjointly S and T where there are defining relators $R_1 \equiv SG^{-1}$ and $R_2 \equiv TH^{-1}$ with $l(S) > 7/8\ l(R_1)$ and $l(T) > 7/8\ l(R_2)$. We also assume that S and T are as large as possible. Except for symmetry, there are only two ways that S and T can lie in a cyclic transform of W:

Case I. $P \subset S$ and $Q \subset T$. Then $l(P) \leqq l(S) \leqq l(R_1) \leqq \max l(R_i)$.

Case II. $X \subset S$ and $Y \subset T$, as indicated in Figure 1. Say $P \equiv E_2 E E_1$, $Q \equiv F_1 F F_2$, $S \equiv E_1 X F_1$ and $T \equiv F_2 Y E_2$. Replacing S by G and T by H in a cyclic transform of W, we obtain $V \equiv EGFH = 1$. We note that the maximality of S implies $E \wedge G$. (Otherwise, if $E \equiv E'a$ and $G \equiv a^{-1}G'$ then we could have chosen $S' \equiv aS$ in place of S.) Similarly, $G \wedge F$, $F \wedge H$ and $H \wedge E$. However, applying Greendlinger's lemma to V, we see that V must freely reduce to the empty word because E, $F < 2/8\ R$ and G, $H < 1/8\ R$. Accordingly, either E, F, G or H must be empty. But PXQ, $QYP < 8/8\ R$; hence S, $T < 8/8\ R$ and therefore G and H are not empty words. Thus E or F is empty. Now $P < 2/8\ R$ implies $l(E_1) < 2/8\ l(R_1)$ and $l(E_2) < 2/8\ l(R_2)$. Consequently, if $E \equiv 1$ then $P \equiv E_2 E_1$ and

$$l(P) = l(E_1) + l(E_2) < 2/8[l(R_1) + l(R_2)] < \max l(R_i).$$

On the other hand, suppose $F \equiv 1$. Then $V \equiv EGH$ and, since V freely reduces to the empty word,

$$l(E) \leqq l(G) + l(H) < 1/8[l(R_1) + l(R_2)].$$

Furthermore, since $P \equiv E_2 E E_1$,

$$l(P) = l(E) + l(E_1) + l(E_2) < 3/8[l(R_1) + l(R_2)] < \max l(R_i).$$

Thus the lemma is proved.

LEMMA 4. *Let h be a basic element in an eighth-group A, and suppose h does not commute with an element $x \in A$. If $W \equiv h^m x h^s y = 1$ then*

$$|m| \leqq l(x) + l(y) + 2 + \max l(R_i).$$

PROOF. We need only consider the case $m > l(x)+2$. If W is freely equal to 1, then by Corollary 1 in [8] we have $|m| \leqq l(x)+l(y)+2$.

On the other hand, suppose $W \not\approx 1$, and say W freely cyclically reduces to $V \equiv PXQY$ where $P \subset h^m$, $X \subset x$, $Q \subset h^s$ and $Y \subset y$. By Lemma 2, h^m, $h^s < 2/8\ R$; hence P, $Q < 2/8\ R$. Also, X, $Y \leqq 4/8\ R$. By Lemma 3, X and Y are nonempty words. Accordingly,

$$l(P) \geqq l(h^m) - l(x) - l(y).$$

But by Lemma 3, $l(P) \leqq \max l(R_i)$; hence

$$|m| \leqq l(h^m) \leqq l(P) + l(x) + l(y) \leqq l(x) + l(y) + \max l(R_i).$$

Thus the lemma is proved.

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