CLUSTER SETS OF MEROMORPHIC FUNCTIONS

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Let $w=f(z)$ be a function meromorphic in the open unit disc D, and let Γ denote the unit circle. Let $C(f, 1)$, $C_\Gamma(f, 1)$, $R(f, 1)$ and $A(f, 1)$ denote, respectively, the cluster set, the boundary cluster set, the range of values and the asymptotic set (set of asymptotic values) of f at 1 (for the terminology see [**3**]). In this note we set

$$C = C(f, 1), \quad C_\Gamma = C_\Gamma(f, 1), \quad R = R(f, 1), \quad A = A(f, 1),$$

and we let W denote the extended w-plane. Classical theorems of W. Gross and F. Iversen state that the boundary of C is contained in C_Γ, and any point of the open set $C-C_\Gamma$ that is not in R is in A; moreover, R covers $C-C_\Gamma$ with the possible exception of at most two points, and if there are two exceptional points, then R is W minus these two points (see [**1**]).

Suppose that C_Γ is not connected. Since C_Γ has exactly two components, there is exactly one multiply connected component of $W-C_\Gamma$, and it is doubly connected. Let U denote the multiply connected component of $W-C_\Gamma$. Clearly $U\subset C$, because C is connected, $C_\Gamma\subset C$ and the boundary of C is contained in C_Γ. Hence $U-R\subset A$.

Our purpose is to establish the following four theorems. Note that the second theorem follows from the first, and the third follows from the first and the second.

THEOREM 1. *If $A\cap U\neq\emptyset$, then $W-R\subset A$.*

THEOREM 2. *Either $U\subset R$ or $W-R\subset A$.*

THEOREM 3. *If $W-R$ contains at least three points, then $U\subset R-A$.*

THEOREM 4. *There exists a normal holomorphic f such that $\infty\in U$.*

REMARK 1. It follows from Theorem 2 that if f is normal, or more generally if f has at most one asymptotic value at 1, and if $a\in U-R$, then $R=W-\{a\}$.

REMARK 2. Theorems 3 and 4 answer a question posed to me by F. Bagemihl.

PROOF OF THEOREM 1. Suppose that $a\in A\cap U$, and let J be a curve in D such that $J\cup\{-1, 1\}$ is a Jordan arc, and f has the limit a on J at 1. Let D_1 and D_2 be the components of $D-J$, and let Γ_j denote the boundary of D_j $(j=1, 2)$. Let the components C_Γ^1 and C_Γ^2 of C_Γ be denoted so that

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$$C_{\Gamma_j}(f, 1) = C_\Gamma^j \cup \{a\} \qquad (j = 1, 2),$$

and let U_j denote the multiply connected component of $W - C_{\Gamma_j}(f, 1)$ (for the notation see [3]). Then, as we observed,

$$U_j \subset C_{D_j}(f, 1), \qquad U_j - R_{D_j}(f, 1) \subset A_{D_j}(f, 1) \quad (j = 1, 2).$$

But $U_1 \cup U_2 = W - \{a\}$, and it follows that $W - R \subset A$.

PROOF OF THEOREM 4. Let S be the universal covering surface of $W - \{2i, \infty\}$, which is the Riemann surface of $g(z') = \exp z' + 2i$. Let L_1 be any component of the set of points of S over the interval $\{w \geqq 1\}$ on the real axis, and let P_1 be the point of L_1 over 1. Let L_2 be the component containing P_1 of the set of points of S over $\{1 + iv : 0 \leqq v \leqq 2\}$, and let P_2 be the point of L_2 over $1 + 2i$. The set of points of S over the circle $\{|w - 2i| = 1\}$ is a single spiral which is divided into two parts by P_2. Let L_3 be the one of these parts such that $L = L_1 \cup L_2 \cup L_3$ has a tangent at P_2. Let S_1 be the component of $S - L$ that is to the right of a point moving on L from P_1 to P_2. Let the sets in the z'-plane that correspond under $w = g(z')$ to L, L_1 and S_1 be denoted L', L_1' and S_1', respectively. Note that L' tends at both ends to ∞, and let $z' = h(z)$ be a conformal mapping of the upper half-disc

$$D^* = D \cap \{z : y > 0\} \qquad (z = x + iy)$$

onto S_1' such that $\{x : -1 \leqq x < 1\}$ corresponds to L_1'. Let $f(z) = g(h(z))$, and continue f to a holomorphic function in all of D by reflection. Clearly

$$C_\Gamma = \{ |w - 2i| = 1\} \cup \{ |w + 2i| = 1\}.$$

We suppose that f is not normal, and we derive a contradiction. By a theorem of Lappan [2, p. 45, Corollary 2], some sequence $\{z_n\} \subset D$ has the property that for any complex number w there exists a sequence $\{z_n'\} \subset D$ such that $\rho(z_n, z_n') \to 0$ (ρ is the hyperbolic metric in D) and $f(z_n') \to w$. (I am indebted to Professor Bagemihl for this reference.) It follows that $z_n \to 1$. Since f is bounded away from $2i$ in the upper half-disc D^* and has the radial limit ∞ at 1, it follows easily that f has ∞ as an angular limit at 1. Hence any Stolz angle at 1 can contain only finitely many z_n. We consider the case where a subsequence of $\{z_n\}$ is contained in D^*, and we suppose, without loss of generality, that $\{z_n\} \subset D^*$. Then $z_n \to 1$ tangentially. Choose $\{z_n'\} \subset D$ such that $\rho(z_n, z_n') \to 0$ and $f(z_n') \to 2i$. Then $z_n' \in D^*$ for sufficiently large n, and this is a contradiction. The proof of Theorem 4 is complete.

REMARK 3. A simple example of a bounded f has as its Riemann surface the universal covering surface of the annulus $\{1<|w|<2\}$. We now construct an f such that $U-R$ contains two points. Let $g(z)$ be a function holomorphic on $\{|z|\leqq 1\}-\{1\}$ such that $g(e^{i\theta})\to 1$ as $\theta\to 0$ from the right and $g(e^{i\theta})\to -1$ as $\theta\to 0$ from the left, and let $f(z)=\exp g(z)$.

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