ALMOST RECURSIVE SETS

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1. If A is any set (a subset of N = the set of nonnegative integers), by ρ_A we denote the function

$$(1.1) \qquad \rho_A(x) = \text{the cardinality of } A_x = \{y \in A \mid y < x\}.$$

A set A is recursive iff (if and only if) ρ_A is a recursive function.

We shall call a set A *almost recursive* iff $\rho_A | A$ (ρ_A, restricted to A) can be extended to a p.r. (partial recursive) function, or, intuitively, iff for every $x \in A$ we can effectively find the number of elements of A which are less than A.

More generally, for any two sets A and B, we shall say that A is *B-almost recursive* iff $\rho_A | B$ can be extended to a p.r. function. In this way, a set A is almost recursive *iff* it is A-almost recursive.

In this paper we prove some theorems on almost recursive sets, in particular, theorems relating those sets to retraceable sets (Dekker and Myhill [1]).

2. If A is a set, by p_A we denote the partial function, with A as domain and range, which maps Min A (the minimal element of A) onto itself, and every other element of A onto the next smaller element of A. The set A is *retraceable* iff p_A can be extended to a p.r. function, which is then called a *retracing function* for A.

If A is an infinite set, by h_A we denote its *principal function*, i.e. the strictly increasing function ranging over A. For such a set, $\rho_A(x) = h_A^{-1}(x)$ (h^{-1} = the inverse function of h), and $p_A(x) = h_A(h_A^{-1}(x) \dot{-} 1)$, for every $x \in A$.

Let us call a 1-1 function f *almost recursive*, iff f^{-1} can be extended to a p.r. function. Then the content of the previous paragraph can be expressed as follows: An infinite set is almost recursive iff its principal function is almost recursive; a p.r. function φ is the retracing function for at least one infinite retraceable set iff there is a strictly increasing almost recursive function f, such that $\varphi(x) = f(f^{-1}(x) \dot{-} 1)$ for all $x \in R_f$ (R_f = the range of f; D_f = the domain of f).

The following theorem is a direct consequence of Proposition 1 of [1]:

THEOREM 2.1. (a) *Every retraceable set is almost recursive.* (b). *Every almost recursive set with a r.e. complement is retraceable.*

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As a consequence of Theorem 2.1 and the fundamental theorems of [1], we obtain

COROLLARY 2.1.1. (a) *Every degree is almost recursive.* (b) *Every r.e. degree is the degree of a r.e. set whose complement is almost recursive.* (c) *Every almost recursive set with r.e. complement is either recursive or hyperimmune.* (d) *There exist immune sets which are not almost recursive, but have r.e. complements.*

PROOF. Theorem 2.1 and: for (a) Theorem T.2 of [1]; for (b) T.3 of [1]; for (c) T.4 of [1]; for (d) the Corollary of T.4 of [1].

Dekker and Myhill have shown that a retraceable set is either recursive or immune, and that there is a continuum of immune retraceable sets. In contrast to this, we have

THEOREM 2.2. (a) *Every recursive set is almost recursive.* (b) *There are continuum many almost recursive sets which are immune.* (c) *There are continuum many almost recursive sets which are unions of an immune set and of an infinite r.e. set.* (d) *There are continuum many almost recursive sets which are productive.*

PROOF. (a) is trivial. (b) follows from Theorem 2.1 and the remark before Theorem 2.2. To prove (c) let h be the principal function of a retraceable immune set such that $10^i<h(i)<10^{i+1}$ for all $i\in N$. (See [1], Theorem T.5.) Define f by $f(2i)=h(i)$ and $f(2i+1)=10^{i+1}$. If $A=R_f$, A is almost recursive, satisfying (c). (By a remark of the referee, one can prove (c) also as follows: put a tree of c retraceable sets inside the even integers and take the union of each branch with the set of odd integers.) For (d), let α be the characteristic function of a productive set E ($\alpha(i)=0\leftrightarrow i\in E$). Define the almost recursive function a by $a(i)=10^i+1+\alpha(i)$. (Then, $a^{-1}(x)=\mu_i(10^i<x<10^{i+1})$ for all $x\in R_a$.) Let $A=R_a$. A is almost recursive and productive, as E is reducible to A ($x\in E\leftrightarrow 10^x+1\in A$).

If F is a (partial) function, then by $\overline{D}_x$, for every $x\in D_F$, we denote the set

$$\overline{D}_x = \{y \in D_F \mid F(y) + 1 = F(x)\}. \tag{2.1}$$

An almost recursive function f will be called *special* iff there is a p.r. extension F of f^{-1} and a p.r. function φ, called the *choice function* for f, such that

$$x \in R_f \wedge R_f \cap \overline{D}_x \neq \varnothing \rightarrow \varphi(x) \text{ defined} \wedge \varphi(x) \in R_f \cap \overline{D}_x, \tag{2.2}$$

where $\overline{D}_x$ is as in (2.1).

THEOREM 2.3. *An infinite almost recursive set is retraceable iff its principal function is special.*

PROOF. Let $A=\{h_A(i)\mid i\in N\}$ be retraceable. Let H be any p.r. extension of $h_A{}^{-1}$ (which exists, as A is also almost recursive) and let p be any p.r. extension of p_A. (Note that $D_p\supset A$.) Define the p.r. extension F of $h_A{}^{-1}$ by $F=H\mid D_p$. Let, for every $x\in D_F$, $\overline{D}_x$ be as in (2.1).

If $x\in A$, $\overline{D}_x\cap A\neq\varnothing$ and $F(x)>0$, then $p(x)=p_A(x)\in\overline{D}_x\cap A$ (as $p_A(x)\in A$ and $F(p_A(x))+1=F(x)$), which shows that p can be used as a choice function for h_A.

Conversely, let h_A be special and let φ be a p.r. choice function for h_A. Let F be the (existing) p.r. extension of $h_A{}^{-1}$, as required by the definition of a special function. Define p as follows:

$$\begin{aligned} p(x) &= \text{Min } A \quad \text{if } F(x) = 0,\\ &= \varphi(x) \qquad \text{if } F(x) > 0. \end{aligned}$$

If $x\in A$ and $\overline{D}_x\cap A\neq\varnothing$, then $\overline{D}_x\cap A=\{p_A(x)\}$; if $x>\text{Min } A$, then $F(x)>0$, if $x=\text{Min } A$, then $F(x)=0$. This proves that p is a *p.r.* extension of p_A, i.e. that A is retraceable.

COROLLARY 2.3.1. *A p.r. function φ is a retracing function for at least one infinite retraceable set iff it is the choice function for at least one special, strictly increasing, function f, such that $\varphi(f(0))=f(0)$.*

We shall say that a *p.r.* function F is *strict* iff there is a strictly increasing function f such that $f^{-1}=F\mid R_f$. Obviously, if F is strict there is an infinite almost recursive set $A=R_f$ such that $\rho_A\mid A=F\mid A$.

Moreover, if D is the domain of a (partial) function F, we define the double sequence $D_i^{(n)}$ of subsets of D by:

$$D_i^{(0)} = \{y\in D\mid F(y)=i\},$$
$$D_i^{(n+1)} = \{y\in D_i^{(n)}\mid \bigvee_z z\in D_{i+1}^{(n)}\wedge z>y\}.$$

With this, we have

THEOREM 2.4. *A p.r. function F, with the domain D, is strict, iff all the sets $H_i=\bigcap_{n=0}^{\infty}D_i^{(n)}$, $i\in N$, are nonempty.*

PROOF. Let $H_i\neq\varnothing$ for all $i\in N$. A number h_0 is in H_0 if there is a number h_1 in H_1 such that $h_0<h_1$ and there is a number h_2 in H_2 such that $h_1<h_2,\cdots$ and so on, ad infinitum. If $H=H_0\times H_1\times H_2\cdots$, then there is an element h of H which is a strictly increasing function, such that $h^{-1}=F\mid R_h$.

Conversely, if there is a strictly increasing h such that $F|R_h=h^{-1}$, then $h(i)\in H_i$ for all $i\in N$.

3. Obviously, an almost recursive set is recursive iff it is recursively enumerable. Also, as the composition of two almost recursive (strictly increasing) functions is almost recursive, we conclude: every infinite almost recursive set contains a continuum of different almost recursive subsets.

Let us note that every infinite almost recursive set A contains a continuum of retraceable sets. To prove this, it is enough to show that A contains at least one infinite retraceable set. So, let $A=\{h_A(i)|i\in N\}$ be almost recursive and infinite. Define h by $h(0)=h_A(0)$ and $h(i+1)=h_A(h(i))$. Then $B=R_h$ is an infinite retraceable subset of A, as $p_B(x)=h_A^{-1}(x)$ for all $x\in B$ which are not the minimal element of B.

The notion of relative almost recursiveness arises naturally in the study of unions and intersections of almost recursive sets.

THEOREM 3.1. *Let A and B be separable almost recursive sets. Then $A\cup B$ is almost recursive iff A is B-almost recursive and B is A-almost recursive.*

PROOF. ("Separable" means: there are two disjoint r.e. sets E_1 and E_2 such that $A\subset E_1$ and $B\subset E_2$.) Let F_1 and F_2 be p.r. functions, such that $\rho_A|A=F_1|A$ and $\rho_B|B=F_2|B$.

Suppose first that $A\cup B$ is almost recursive and let F_3 be a p.r. function such that $\rho_{A\cup B}|A\cup B=F_3|A\cup B$. As, for every $x\in A$, we have $\rho_{A\cup B}(x)=\rho_A(x)+\rho_B(x)$, we have especially, for all $x\in A$, that $\rho_B(x)=F_3(x)-F_1(x)$. This proves that B is A-almost recursive. By symmetry, A is also B-almost recursive.

Let now $\rho_A|B=F_4|B$ and $\rho_B|A=F_5|A$, where F_4 and F_5 are p.r. functions. Define ϕ by

$$\begin{aligned}\phi(x) &= F_1(x)+F_5(x) \quad \text{if } x\in E_1\\ &= F_2(x)+F_4(x) \quad \text{if } x\in E_2.\end{aligned}$$

As easily seen, $\rho_{A\cup B}|A\cup B=\phi|A\cup B$, i.e. $A\cup B$ is almost recursive.

It is not difficult to give examples of disjoint almost recursive sets whose union is not almost recursive. If α is the characteristic function of a nonrecursive set, define the function a by $a(i)=10^i-1$ if $\alpha(i)=0$, and $a(i)=10^i+1$ if $\alpha(i)=1$. Let $b(i)=10^i$. If A is the range of a and B the range of b, it is not difficult to show that both sets are almost recursive, but that their union is not almost recursive.

In the proof of Theorem T.5 of [1], Dekker and Myhill give examples of two separable retraceable sets A and B such that $A\cup B$ is not retraceable. However, we could not use their example above, as for their sets A and B the union $A\cup B$ is almost recursive. In general, if a is the principal function of a retraceable immune set A, for which there is a strictly increasing recursive function f, such that for all $i\in N$, $f(i)<a(i)<f(i+1)$, and if B is the range of $b(i)=f(a(i))$ (B is then retraceable) the union $A\cup B$ will not be retraceable, but it will be almost recursive. In view of this, the following theorem, suggested by the referee, has importance of its own.

THEOREM 3.2. *There exist disjoint, recursively separable retraceable sets A and B, whose complements are recursively enumerable, such that $A\cup B$ is not almost recursive.*

PROOF. Let A_0 be a co-r.e. retraceable set of Turing degree $0'$; and let B_0 be a co-r.e. retraceable set of degree $\boldsymbol{b}$ where $0\lneqq\boldsymbol{b}\lneqq 0'$. (We here use Dekker's result that every r.e. degree contains an r.e. set with retraceable complement.) Let $A=\{2x\mid x\in A_0\}$, $B=\{2x+1\mid x\in B_0\}$. Then $\overline{A}$, $\overline{B}$ are clearly recursively separable, co-r.e., and retraceable. Also, $[A\cup B]^-$ is co-r.e., so it would be retraceable if it were almost recursive. But retraceable sets are introreducible, and $[A\cup B]^-$ is obviously not introreducible. This finishes the proof.

(*Comment*: the bulk of the above argument is due to Dekker, and was used by him in his paper "The minimum of two regressive isols" [Math. Z. **83** (1964), 345–366], to show that the class Λ_R of regressive isols is not additively closed. Obviously, to eliminate use of a theorem on degrees, we can replace B_0, in the above, by an infinite *recursive* set.)

The intersection of any two retraceable sets is retraceable. However, for almost recursive sets, the situation is not so simple, as seen from

THEOREM 3.3. *Let A and B be almost recursive sets, such that $C=A\cap B$ is infinite.*

Then, C is almost recursive iff at least one of the sets $A-B$ and $B-A$ is C-almost recursive (in which case the other one is also C-almost recursive).

PROOF. Let F_1 and F_2 be p.r. functions, such that $\rho_A|A=F_1|A$ and $\rho_B|B=F_2|B$.

Remark, that for all $x\in C$

$$(3.1) \qquad \rho_C(x) = \rho_A(x) - \rho_{A-B}(x) = \rho_B(x) - \rho_{B-A}(x).$$

Let now $A-B$ be C-almost recursive and let F_3 be a p.r. function such that $\rho_{A-B}|C = F_3|C$. Then, for all $x \in C$, $\rho_C(x) = F_1(x) \dot{-} F_3(x)$, which proves that C is almost recursive. As then, for all $x \in C$, $\rho_{B-A}(x) = F_2(x) \dot{-} \{F_1(x) \dot{-} F_3(x)\}$, it follows that $B-A$ is C-almost recursive.

The converse implication is obvious from (3.1).

THEOREM 3.4. *There exists almost recursive sets A and B, such that $C = A \cap B$ is infinite, but not almost recursive.*

PROOF. Let E be a r.e. but not recursive set. Define the function a by $a(i) = 10^i$ if $i \in E$ and $a(i) = 10^i + 1$ if $i \notin E$. a is a strictly increasing function. If $\varphi(x) = \mu_i\,(10^i \leqq x < 10^{i+1})$, then $a^{-1}(x) = \varphi(x)$ for all $x \in R_a$. Therefore, $A = R_a$ is an almost recursive set.

Let $b(i) = 10^i$ and $B = R_b$. The set B is recursive (and, therefore, also almost recursive). Now,

$$C = A \cap B = \{10^i \mid i \in E\}.$$

As

$$x \in C \leftrightarrow \bigvee_{i=0}^{x} (x = 10^i \wedge i \in E)$$

the set C is an infinite r.e. set. If C were almost recursive, it would be recursive, by the remark on the beginning of this section. Then E would also be recursive. Contradiction.

It is obvious that one can pursue the problem of almost recursivity further. For instance, let us say that a one-to-one partial function φ is *partially almost recursive* iff its domain D_φ is a r.e. set and φ^{-1} can be extended to a p.r. function. Let us call ranges of such functions *almost r.e. sets*. Then one can easily prove: an infinite set is almost r.e. iff it is the range of an almost recursive function. (Let us point out that the condition on D_φ cannot be removed, as, without it, every set would be almost r.e.) Moreover, relative almost recursivity suggests the possibility to consider relative retraceability. These and similar problems and possibilities will be postponed for a later paper.

REFERENCE

1. J. C. E. Dekker and J. Myhill, *Retraceable sets*, Canad. J. Math. 10 (1958), 357–378.

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