ON EQUATIONS OF THE WIENER-HOPF TYPE IN SEVERAL COMPLEX VARIABLES

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1. Introduction. In a previous communication [1] we established conditions under which the additive decomposition of a function of several complex variables is unique. In the analysis presented here we use the considerations of [1] to formulate a Wiener-Hopf problem in several complex variables which will be solved for a class of physically important kernels [2] by the method of successive approximations.

Statement of the Wiener-Hopf problem. Let $K(z_1, \ldots, z_n) \times g_1(z_1, \ldots, z_n), z_j=x_j+iy_j$, be analytic in a tube $T: \{-\delta_i < y_i < \delta_i, x_i \in (-\infty, \infty)\}$, and let $K$ have the form

$$K(z_1, \ldots, z_n) = 1 - \lambda H(z_1, \ldots, z_n),$$

where $\lambda$ is a complex parameter which may be made small and $H(z_1, \ldots, z_n)$ is a uniformly bounded analytic function in the tube $T$. Suppose that the $L^2$ norm

$$\|g_1\|_2 = \left\{ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |g_1(z_1, \ldots, z_n)|^2 dx_1 \cdots dx_n \right\}^{1/2}$$

converges in $T$, then $\|Kg_1\|_2$ also converges in $T$. Consequently, there exists in $T$ a unique additive decomposition $Kg_1 = \sum_{i=1}^{2^n} f_i$, where each $f_i$ is analytic and bounded in an octant shaped tube $T_i$ containing the interior of $T$, and $f_i \to 0$ as any one of the $y_i \to \pm \infty$ in $T_i$.

Given $f_1(z_1, \ldots, z_n)$ and the particular kernel $K(z_1, \ldots, z_n)$ in (1) determine the remaining $2^n$ unknown analytic functions $g_1, f_2, \ldots, f_{2^n}$ appearing in the decomposition of $Kg_1$.

2. Solution of the Wiener-Hopf problem. Let the functions $g_1(z_1, \ldots, z_n)$ and $f_1(z_1, \ldots, z_n)$ be assumed to be analytic in the octant shaped tube $T_1: \text{Im}(z_i) > 0, i=1, 2, \ldots$, and to have bounded $L^2$ norms in $T$. The decomposition of $Kg_1$ has the form

$$g_1 - \lambda H g_1 = \sum_{i=1}^{2^n} f_i,$$

and because this decomposition is unique, Cauchy integration yields
\[ g_1(z_1, \ldots, z_n) = f_1(z_1, \ldots, z_n) \]

\[ + \frac{\lambda}{(2\pi i)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{H(\zeta_1, \ldots, \zeta_n)g_1(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \, d\zeta_1 \cdots d\zeta_n, \]

where \( \text{Im}(z_1) > 0, \text{Im}(z_2) > 0, \ldots, \text{Im}(z_n) > 0 \).

Equation (3) may be reduced to a singular integral equation for \( g_1(z_1, \ldots, z_n) \) by allowing \( \text{Im}(z_i) \rightarrow 0^+, \ldots, \text{Im}(z_n) \rightarrow 0^+ \) and introducing the generalized Plemelj formulas \([3]\). The result is

\[ g_1 = f_1 + \frac{\lambda}{2^n} \prod_{i=1}^{2^n} (I + S_i)(Hg_1), \]

where \( I \) is the identity operator and where

\[ S_i = \frac{P}{\pi i} \int_{-\infty}^{\infty} \frac{d\zeta_i}{(\zeta_i - z_i)}, \]

are principal value integral operators. We state our main result for equation (4) as a

**Theorem.** Let the maximum modulus of \( H(z_1, \ldots, z_n) \) on \( \text{Im}(z_i) = 0, \ i = 1, 2, \ldots, n \) be denoted by \( \max |H| \). Then for \( 0 < |\lambda| < (\max |H|)^{-1} \), and \( g_1 \) in the complete normed linear space \( L_2 \),

\[ T(g_1) = f_1 + \frac{\lambda}{2^n} \prod_{i=1}^{2^n} (I + S_i)(Hg_1) \]

is a contraction mapping with respect to the \( L_2 \) norm. Hence the integral equation \( T(g_1) = g_1 \) has one and only one fixed point belonging to \( L_2 \). This fixed point is the limit of a sequence of successive approximations converging in \( L_2 \) norm to \( g_1 \).

**Proof.** Let \( g_1 \) and \( h_1 \) be members of the \( L_2 \) function space and analytic in \( T_1 \), we have

\[ \| T(g_1) - T(h_1) \|_2 = \frac{|\lambda|}{2^n} \left\| \prod_{i=1}^{2^n} (I + S_i) H(g_1 - h_1) \right\|_2. \]

Expanding the operator \( \prod_{i=1}^{2^n} (I + S_i) \) and applying Minkowski's inequality to the result yields

\[ \| T(g_1) - T(h_1) \|_2 \leq |\lambda| \max |H| \| g_1 - h_1 \|_2 \]

where we have made use of the fact that the principal value operator \( S_i \) gives the Hilbert transform, with respect to the \( i \)th variable, of the
function on which it operates and that the Hilbert transform is a bounded linear operator in $L_2$ satisfying $\|S_i H g_1\|_2 = \|H g_1\|_2$ and
\[
\left\| \prod_{i=1}^{2^n} S_i H g_1 \right\|_2 = \|H g_1\|_2.
\]

We have established that $T(g_1)$ is a contraction mapping with respect to the $L_2$ norm provided $|\lambda| \max |H| < 1$. The remainder of the theorem is then a consequence of Banach's fixed point theorem [5].

Since we have now constructed a $g_1$ such that the conditions of Bochner's theorem [1] apply to the left-hand side of (2), the unknowns $f_2, f_3, \ldots, f_{2^n}$ in (2) are now uniquely determined by the Cauchy integral decomposition of $g_1 - \lambda H g_1$. This solves the Wiener-Hopf problem for kernels of the form indicated in (1).

REFERENCES


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