CONVERGENCE OF A SEQUENCE OF POWERS

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A well-known theorem states that if a stochastic matrix (definition below) of finite order has all positive entries in it, then the sequence of its powers (or iterates) converges to a limit; see [3, p. 173]. In this paper we will give a new proof of this result using elementary ideas from the theory of partially ordered linear algebras. Our proof does not use the internal structure of the given matrix; therefore, it can be applied to nonnegative operators.

The basic definition of a partially ordered linear algebra (pola) is as follows. A pola A is first of all a linear algebra with real numbers as scalars. Real numbers will usually be denoted by small Greek letters. Multiplication of elements of A is assumed to be associative, but not necessarily commutative. Next, the linear algebra A is a partially ordered set subject to the following conditions (x, y, z denote arbitrary elements of A and α denotes an arbitrary real number under the specified restrictions in each condition):

(a) if \( x \leq y \), then \( x+z \leq y+z \);
(b) if \( 0 \leq x \) and \( 0 \leq y \), then \( 0 \leq xy \);
(c) if \( 0 \leq \alpha \) and \( 0 \leq x \), then \( 0 \leq \alpha x \);
(d) for any \( x \in A \) there exists \( y \geq 0 \) and \( z \geq 0 \) such that \( x = y - z \).

We may also introduce a form of order completeness described as follows. The pola A is said to be Dedekind \( \sigma \)-complete if it satisfies the following condition: if \( \{x_n\} \) is a sequence of elements from A such that \( x_1 \geq x_2 \geq \cdots \geq 0 \), then \( \inf \{x_n\} \) exists. See [4, pp. 9-11]. Of course, \( \inf \{x_n\} \) denotes the infimum (greatest lower bound) of the sequence \( \{x_n\} \). It is defined as follows: \( \inf \{x_n\} = x \) means that

1. \( x \leq x_n \) for all \( n \);
2. if \( y \leq x_n \) for all \( n \), then \( y \leq x \).

We now introduce a concept of order convergence: a sequence \( \{y_n\} \) of elements from A is said to order converge to \( y \in A \) if and only if there exists a sequence \( \{z_n\} \) of elements from A such that \( z_1 \geq z_2 \geq \cdots \geq 0 \), \( \inf \{z_n\} = 0 \), and \( -z_n \leq y_n - y \leq z_n \) for all \( n \). In this case we write \( \text{o-lim } y_n = y \).

In general, multiplication is not continuous with respect to order convergence; see [2]. We say that multiplication is continuous if the following holds: for every sequence \( \{x_n\} \) such that \( x_1 \geq x_2 \geq \cdots \geq 0 \)
and \( \inf \{ x_n \} = 0 \) and for every \( y \geq 0 \) we have \( \inf \{ x_n y \} = \inf \{ y x_n \} = 0 \).

The reader may find more basic information on partially ordered sets, etc., in [1] and [5].

If \( m \) is a fixed positive integer and if \( A \) denotes the real linear algebra of all matrices of order \( m \) with real entries, then \( A \) can be regarded as a pola as follows. If \( x \in A \) and \( y \in A \), where \( x = [\alpha_{ij}] \) and \( y = [\beta_{ij}] \), then \( x \preceq y \) means that \( \alpha_{ij} \leq \beta_{ij} \) for all \( i, j \). It is easy to show that in \( A \) multiplication is continuous. A stochastic matrix \([\alpha_{ij}]\) is one such that \( \alpha_{ij} \geq 0 \) for all \( i, j, = 1, \cdots, m \) and \( \sum_{i=1}^{m} \alpha_{ij} = 1 \) for all \( i = 1, \cdots, m \). Now suppose \( x = [\alpha_{ij}] \) is a stochastic matrix with \( \alpha_{ij} \geq \delta > 0 \) for all \( i, j \). It is easily seen that \( 0 \leq x^n \leq \delta^{-1}x \) for all \( n \). It turns out that this is all that is needed to prove that \( \lim x^n \) exists. One can easily construct other kinds of nonnegative matrices which satisfy this condition. By referring to [2] the reader will see these ideas can be applied to bounded operators on a real Banach space. We now prove the main theorem.

**Theorem.** Let \( A \) be a partially ordered linear algebra which is Dedekind \( \sigma \)-complete. If \( x \in A \) and if for some \( \delta \geq 1 \) we have \( 0 \leq x^n \leq \beta x \) for all \( n = 1, 2, \cdots \), then \( \lim x^n = u \) exists. Also, \( 0 \leq u^2 \leq u \). If, in addition, we assume that multiplication is continuous, then \( u = u^2 = xu = ux \).

**Proof.** We begin by defining \( \lambda_1 = \beta \) and then by induction \( \lambda_{n+1} = \lambda_n (1 + \lambda_1) (\lambda_1 + \lambda_n)^{-1} \) for all \( n = 1, 2, \cdots \). This latter expression can be rewritten \( \lambda_{n+1} = 1 + \lambda_1 (\lambda_n - 1) (\lambda_1 + \lambda_n)^{-1} \) which means that since \( \lambda_1 = \beta \geq 1 \), we have \( \lambda_n \geq 1 \) for all \( n \). Consequently, we see that \( 0 \leq \lambda_{n+1} - 1 \leq \lambda_1 (1 + \lambda_1)^{-1} (\lambda_n - 1) \) for all \( n \). If we put \( \alpha = \beta (1 + \beta)^{-1} < 1 \), then we can show by induction that \( \lambda_n - 1 \leq \alpha^{n-1} (\beta - 1) \) for all \( n \).

We now show by induction that for each \( n \) we have \( x^k \leq \lambda_n x^n \) for all \( k \geq n \). The assumption in our theorem states that this is true if \( n = 1 \). Now suppose that for some \( n = p \geq 1 \) we have \( x^k \leq \lambda_p x^p \) for all \( k \geq p \). Take any \( q \geq p \) and define \( r = q + 1 - p \geq 1 \). Now note that \( 0 \leq (\lambda_1 x - x^r) (\lambda_p x^p - x^q) \) or \( (\lambda_1 + \lambda_p) x^{q+1} \leq \lambda_1 \lambda_p x^{p+1} + x^{q+1} \), which is obtained from the previous inequality after multiplying and using the fact that \( r + p = q + 1 \). Now since \( q + r - 1 \geq p \), we see that \( x^{q+r-1} \leq \lambda_p x^p \) which means that \( x^{q+r} \leq \lambda_p x^{p+1} \). Therefore,

\[
(\lambda_1 + \lambda_p) x^{q+1} \leq (\lambda_1 \lambda_p + \lambda_p) x^{p+1},
\]

which means that \( x^{q+1} \leq \lambda_{p+1} x^{p+1} \) for all \( q + 1 \geq p + 1 \). This completes the proof by induction.

Now let us define \( z_n = \mu x^n \) and \( y_n = x^n + z_n \), where \( \mu = (\beta - 1) (1 + \beta)^2 \). It is clear that \( z_1 \geq z_2 \geq \cdots \geq 0 \) and \( \inf \{ z_n \} = 0 \). We note that
\[ z_n - z_{n+1} = \alpha^{n-1}\beta(\beta - 1)x, \] is easily be computed by recalling that \( \alpha = \beta(1 + \beta)^{-1}. \) Now \( 0 \leq \lambda_n x^n - x^{n+1} = x^n - x^{n+1} + (\lambda_n - 1)x^n \leq x^n - x^{n+1} + \alpha^{n-1}(\beta - 1)\beta x = x^n - x^{n+1} + z_n - z_{n+1} = y_n - y_{n+1}. \) Consequently, \( y_1 \geq y_2 \geq \cdots \geq 0. \) Since \( A \) is Dedekind \( \sigma \)-complete, we know that
\[ u = \inf \{ y_n \} \text{ exists}. \] It is easy to show that \( -z_n \leq x^n - u \leq y_n - u \) for all \( n. \) Since \( \inf \{ y_n - u + z_n \} = 0, \) we have that \( \lim x^n = u. \)

It is easily seen that \( xu \leq xy_n = x^{n+1} + xz_n \leq x^{n+1} + (1 + \beta)z_{n+1} = y_{n+1} + \beta z_{n+1} \) for all \( n. \) Thus, \( xu \leq u. \) From this it follows that \( x^n u \leq u \) for all \( n. \) Hence it follows that \( u^2 \leq x^n u = x^n u + z_n u \leq u + \mu \alpha^n u \) for all \( n. \) Hence \( u^2 \leq u. \)

Now let us assume that multiplication is continuous. Since \( 0 \leq xy_n - xu = x(y_n - u) \) and since \( \inf \{ x(y_n - u) \} = 0, \) we see that \( \inf \{ xy_n \} = xu. \) It is clear that \( xy_n \geq x^{n+1}, \) which means that \( xy_n + z_n \geq y_{n+1} \geq u \) for all \( n. \) Since \( \inf \{ xy_n + z_n \} = xu, \) we see that \( xu \geq u. \) We have already shown that \( xu \leq u. \) Hence, \( xu = u. \) Similarly, we can show that \( ux = u. \)

Now \( 0 \leq uy_n - u^2 = u(y_n - u). \) Since \( \inf \{ u(y_n - u) \} = 0, \) we see that \( \inf \{ uy_n \} = u^2. \) From what was just proved above we see that \( uy_n = u + \mu \alpha^n u \) for all \( n. \) Hence, \( u^2 = \inf \{ uy_n \} = u. \)

REFERENCES


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