ELEMENTS OF A NORMED ALGEBRA WHOSE 2^nth POWERS LIE CLOSE TO THE IDENTITY

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In [1], R. H. Cox stated that if x is a square matrix all of whose positive powers lie within a distance $\alpha<1$ of the identity matrix 1, then $x=1$. Nakamura and Yoshida [3] extended this result to bounded operators on Hilbert space; their argument used the mean ergodic theorem. Hirschfeld [2] recently presented a proof, based on spectral theory, showing that the result is valid for elements of any normed algebra. Actually this had been established earlier by Wallen [4], who gave a concise and elementary argument using a significantly weakened hypothesis: he required only that $\|x^n-1\|=o(n)$ and that $\liminf n^{-1}(\|x-1\|+\|x^2-1\|+\cdots+\|x^n-1\|)<1$. (Wils [5] also deals with this subject matter; I thank the referee for calling this reference to my attention.)

In this note we examine a different weakening of Cox's hypotheses: we impose conditions only on the 2^nth powers of x.

THEOREM. *Let x be an element of a normed algebra with identity* 1. *Let $\alpha_n=\|x^{2^n}-1\|$. Suppose that* $\limsup \alpha_n<1$. *Then for some n, $x^{2^n}=1$. Assume that in addition every $\alpha_k<2$. Then $x=1$.*

PROOF. We write $x^{2^n}=1+y_n$. Then $(1+y_n)^2=1+y_{n+1}$ so that $2y_n=y_{n+1}-y_n^2$, whence

$$2\alpha_n \leqq \alpha_{n+1}+\alpha_n^2. \tag{1}$$

Define $\beta_n=\sup_{k\geqq n}\alpha_k$. Then (1) implies

$$2\alpha_n \leqq \beta_{n+1}+\beta_n^2.$$

Hence if $k\geqq n$ we have

$$2\alpha_k \leqq \beta_{k+1}+\beta_k^2 \leqq \beta_{n+1}+\beta_n^2.$$

Taking the supremum over all $k\geqq n$, it follows that

$$2\beta_n \leqq \beta_{n+1}+\beta_n^2. \tag{2}$$

Because $\beta_{n+1}\leqq\beta_n$, this implies

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$$\beta_n \leqq \beta_n^2. \tag{3}$$

Now by assumption $\lim \beta_n < 1$. Let n be the smallest integer such that $\beta_n < 1$. Then (3) implies that $\beta_n = 0$, whence $\alpha_n = 0$, i.e. $x^{2^n} = 1$.

Suppose that this n is greater than 0. Then (1), applied to α_{n-1}, says $2\,\alpha_{n-1} \leqq \alpha_{n-1}^2$, i.e. $\alpha_{n-1} \geqq 2$. Hence if every $\alpha_k < 2$ we must have $n = 0$ and $x = 1$.

References

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