SUBALGEBRAS OF GROUP ALGEBRAS

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I. Let G be a locally compact group and m its Haar measure. For any m-measurable subset S of G, let $L(S)$ be the subspace of $L^1(G)$ consisting of elements f such that $\int_{G\setminus S} |f| \, dm = 0$. If S is a subsemigroup then $L(S)$ is a subalgebra of $L^1(G)$. Various papers ([**4**], [**5**] and [**7**]) have been devoted to the study of $L(S)$ and to the question of whether there is a subsemigroup T such that $L(S) = L(T)$ whenever $L(S)$ is an algebra. In [**5**] it is shown that this is the case whenever S is contained in a σ-compact subset. A related problem is the following. Let dS be the set of all x in G such that each measurable neighborhood of x meets S in a set of positive measure. Whenever $L(S)$ is an algebra, dS is a subsemigroup [**7**], but it need not be true that $L(S) = L(dS)$. In this paper we show that in certain cases $L(S) = L(dS)$. Using this we give very easy proofs of some of the results in [**4**] and [**7**].

Let $M(G)$ be the Banach $*$-algebra of bounded regular Borel measures on G. (We follow [**3**] in the definition of Borel subsets etc.) For a Borel subset S of G, let $M(S)$ be the set of $\mu \in M(G)$ with $|\mu|(G\setminus S) = 0$. Suppose that S is a measurable subsemigroup of G so that $L(S)$ is a subalgebra of $L^1(G)$. Let $L(S)^\pi$ be the algebra of left multipliers of $L(S)$, i.e. the algebra of bounded linear maps π of $L(S)$ into itself such that $\pi(f * g) = \pi f * g$. Let $S^\pi = \{x \in G: xS\setminus S \text{ is locally null}\}$. In this paper we show that if G is abelian then S^π is closed and that $M(S^\pi) \subset L(S)^\pi$, and under certain additional hypotheses $M(S^\pi) = L(S)^\pi$. This question has been considered by Birtel in [**1**] and T. A. Davis in [**2**]. The question is of some interest since Davis (op. cit.) showed that for abelian G, the Wiener-Pitt phenomenon occurs for measures in $L(S)^\pi$.

II. Let $C_0(G)$ be the Banach space of continuous complex-valued functions on G which "vanish at infinity." Let f be a function on G; the support of f, Supp(f) is the closure of $\{x \in G: f(x) \neq 0\}$. Let $K(G)$ be the subspace of $C_0(G)$ of functions whose support is compact. Let μ be a measure on G. The support of μ, Supp(μ) is the smallest closed subset of G whose complement has $|\mu|$-measure zero.

It is well known that $L^1(G)$ may be identified with the subspace of $M(G)$ of measures that are absolutely continuous with respect to the Haar measure m, and as such is a two-sided closed ideal in $M(G)$. The so-topology (resp. so_r-topology) on $M(G)$ is the coarsest topology

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such that for every λ in $L^1(G)$ the mapping $\mu \mapsto \mu * \lambda$ (resp. $\mu \mapsto \lambda * \mu$) is a continuous mapping of $M(G)$ into $L^1(G)$.

The first proposition is a well-known property of the support of a measure, and we omit the proof.

PROPOSITION 1. *Let S be a closed subset of G and let $\mu \in M(G)$. Then μ is in $M(S)$ if and only if $\mu(f) = 0$ for each $f \in K(G)$ with $S \cap \mathrm{Supp}(f) = \varnothing$.*

For $x \in G$, let ϵ_x be the Dirac measure at x, i.e. $\epsilon_x(f) = f(x)$ for $f \in C_0(G)$.

PROPOSITION 2. *Let S be a Borel subset of G. S is closed if and only if $M(S)$ is so-closed.*

PROOF. Suppose that $M(S)$ is *so*-closed and that a is in the closure of S. The mapping $x \mapsto \epsilon_x$ is *so*-continuous [**3**, 20.4] so that $\epsilon_a \in M(S)$. This means that $\epsilon_a(G \backslash S) = 0$ and consequently $a \in S$. Thus S is closed.

Now suppose that S is closed and let $\mu \in \mathrm{Cl}_{so}\, M(S)$. There is a net $(\mu_j : j \in J)$ in $M(S)$ such that $\mu_j \xrightarrow{so} \mu$, and $\mathrm{Supp}(\mu_j) \subset S$. Let f in $K(G)$ be such that $S \cap \mathrm{Supp}(f) = \varnothing$. There is a symmetric neighborhood V of e in G such that $S \cap \mathrm{Supp}(f)\, V = \varnothing$ [**3**, 4.10]. Let $\epsilon > 0$ be given, then by the uniform continuity of f there is a neighborhood U of e in G such that $|f(xy) - f(x)| \leqq \epsilon/2\|\mu\|$ for all x in G and y in U. Let λ be a positive measure in $L^1(G)$ such that $\mathrm{Supp}(\lambda) \subset U \cap V$, and $\|\lambda\| = 1$. Then $|\int f(xy)d\lambda(y) - f(x)| \leqq \epsilon/2\|\mu\|$ so that $|\mu * \lambda(f) - \mu(f)| \leqq \epsilon/2$. If $y \in V$ and $x \in G$ is such that $f(xy) \neq 0$ then $x \in \mathrm{Supp}(f)\, V^{-1} \subset \mathrm{Supp}(f)\, V$, consequently $\int_V f(xy)d\lambda(y) = 0$ whenever $x \in S$. Thus $\mu_j * \lambda(f) = 0$. Since $\lambda \in L^1(G)$ there is a j_0 such that $j \geqq j_0 \Rightarrow \|\mu_j * \lambda - \mu * \lambda\| \leqq \epsilon/2\|f\|$. Thus for $j \geqq j_0$, we have

$$
\begin{aligned}
|\mu(f)| &\leqq |\mu(f) - \mu * \lambda(f)| + |\mu_j * \lambda(f) - \mu * \lambda(f)| + |\mu_j * \lambda(f)| \\
&\leqq \epsilon/2 + \|\mu_j * \lambda - \mu * \lambda\|\, \|f\| \\
&\leqq \epsilon.
\end{aligned}
$$

Therefore we must have $\mu(f) = 0$ for any $f \in K(G)$ with $S \cap \mathrm{Supp}(f) = \varnothing$. By Proposition 1, μ is in $M(S)$.

REMARK. Let $\sigma(M(G), C_0(G))$ be the coarsest topology on $M(G)$ such that for each $f \in C_0(G)$, the map $\mu \mapsto \mu(f)$ is continuous. It is not difficult to show that a Borel subset S of G is closed if and only if $M(S)$ is $\sigma(M(G), C_0(G))$-closed.

III. Let S be an m-measurable subset of G, define $dS \subset G$ by $dS = \{x \in G$: if U is an open neighborhood of x, then $m(U \cap S) > 0\}$. If S is locally null then $L(S) = (0)$. To avoid this trivial case we shall suppose throughout that S is not locally null and then $dS \neq \emptyset$. Note that dS is always closed and that $L(S) = L(S \cap dS)$ [7].

THEOREM 1. *Let S be an m-measurable subset of G. Then $L(S)$ and $L(dS)$ are dense in $M(dS)_{so}$.*

PROOF. Since dS is closed, $M(dS)$ is *so*-closed (Proposition 2) and since $L(S) \subset L(dS)$ it suffices to show that $M(dS) \subset \mathrm{Cl}_{so} L(S)$. Let $x \in dS$, $\lambda \in L^1(G)$ and $\epsilon > 0$ be given. By [3, 20.15] there is a neighborhood U of e such that $\|\mu * \lambda - \lambda\| \leqq \epsilon$ for every positive measure $\mu \in M(G)$ with $\|\mu\| = 1$ and $\mu(G \backslash U) = 0$. Let $V = xU$, then V is a neighborhood of x. Let $\mu \in L(S \cap V)$ be a positive measure with $\|\mu\| = 1$. Then $\epsilon_{x^{-1}} * \mu\ (G \backslash U) = 0$ and $\|\epsilon_{x^{-1}} * \mu\| = 1$. Therefore

$$\|\mu * \lambda - \epsilon_x * \lambda\| = \|\epsilon_{x-1} * \mu * \lambda - \lambda\| \leqq \epsilon.$$

Clearly $\mu \in L(S)$ so that $\epsilon_x \in \mathrm{Cl}_{so} L(S)$. Thus $x \in dS$ implies $\epsilon_x \in \mathrm{Cl}_{so} L(S)$. By Proposition 2 of [6], each $\lambda \in M(dS)$ is an *so*-adherence point of the linear span of $\{\epsilon_x : x \in dS\}$ so it follows that $M(dS) \subset \mathrm{Cl}_{so} L(S)$.

PROPOSITION 3. *Let S be an m-measurable subset of G. If $L(S)$ is an algebra then $L(dS) * L(dS) \subset L(S)$.*

PROOF. Let $\mu \in L(dS)$ and let $(\mu_j : j \in J) \subset L(S)$ be a net such that $\mu_j \xrightarrow{so} \mu$. Let $\lambda \in L(S)$ then $\mu_j * \lambda \in L(S)$ and $\mu_j * \lambda$ converges in the norm to $\mu * \lambda$. Therefore $\mu * \lambda \in L(S)$ since $L(S)$ is norm closed. This shows that $L(dS) * L(S) \subset L(S)$. Theorem 1 remains true if we replace the *so*-topology by the so_r-topology. Consequently for $\lambda \in L(dS)$ there is a net $(\lambda_j : j \in J)$ in $L(S)$ such that $\lambda_j \xrightarrow{so_r} \lambda$. For $\mu \in L(dS)$ we have $\mu * \lambda_j \in L(S)$ and $\mu * \lambda_j$ converges to $\mu * \lambda$ in the norm. This means that $\mu * \lambda \in L(S)$ which proves the proposition.

COROLLARY 1. *If $L(S)$ is an algebra and if $e \in dS$, then $L(dS) = L(S)$.*

PROOF. This follows from the proposition, since $e \in dS$ implies that $L(dS)$ has an approximate unit.

In [7] Simon showed that if $L(S)$ is a nontrivial $*$-subalgebra then dS is a subgroup of G. Using this and Corollary 1, we have

COROLLARY 2. *Suppose that $L(S)$ is a $*$-subalgebra, then $L(S) = L(dS)$.*

COROLLARY 3. *Suppose that G is compact. If $L(S)$ is a subalgebra of $L^1(G)$ then dS is a closed subgroup of G and $L(S) = L(dS)$.*

PROOF. A closed subsemigroup of a compact group is necessarily a subgroup.

DEFINITION. Let S be a measurable subset of G. If $L(S)$ is an algebra we shall follow [7] and call $L(S)$ a vanishing algebra. A vanishing algebra is called a maximal vanishing algebra if it is a proper subalgebra and if for every vanishing algebra $L(T)$, $L(S)\subset L(T)$ implies $L(S)=L(T)$ or $L(T)=L^1(G)$.

We now give a very easy proof of a result due to Liu [**4**, Theorem 5].

THEOREM 2. *If $L(S)$ is a maximal vanishing algebra, then dS is a closed subsemigroup and $L(S)=L(dS)$.*

PROOF. We need only show $L(S)=L(dS)$. If $L(S)\neq L(dS)$ then by the maximality of $L(S)$, $L(dS)=L^1(G)$, thus $d(dS)=dS$ is a subgroup and by Corollary 1, $L(dS)=L(S)$ a contradiction.

IV. In this section we assume throughout that G is a locally compact abelian group.

PROPOSITION 4. *Let $\pi\in L(S)^\pi$, then given $x\in dS$, there is a measure $\mu\in M(dS)$ such that for each $f\in L(S)$, $\epsilon_x * \pi f=\mu * f$.*

PROOF. Let $(e_j: j\in J)$ be a norm bounded net in $L(S)$ such that $e_j\xrightarrow{so}\epsilon_x$. Then $(\pi e_j: j\in J)$ is a norm bounded net in $M(G)$ and there is therefore a subnet $(\pi e_{j(i)})$ and a $\mu\in M(G)$ such that $\pi e_{j(i)}\xrightarrow{so}\mu$ (here σ is the $\sigma(M(G), C_0(G))$-topology). Since convolution is $\sigma(M(G), C_0(G))$-continuous [6] we have for f in $L(S)$, $\pi e_{j(i)} * f\xrightarrow{\sigma}\mu * f$. On the other hand convolution is also *so*-continuous, so that

$$\pi e_{j(i)} * f = \pi(e_{j(i)} * f) = e_{j(i)} * \pi f \xrightarrow{so} \epsilon_x * \pi f.$$

Thus we have $\epsilon_x * \pi f=\mu * f$. To complete the proof we need to show that $\mu\in M(dS)$, and this follows since $M(dS)$ is $\sigma(M(G), C_0(G))$-closed (remark following Proposition 2) and $L(S)\subset M(dS)$.

PROPOSITION 5. *Let S be a subsemigroup of G with nonzero Haar measure and let $S^\pi=\{x\in G: xS\backslash S$ is locally null$\}$. Then S^π is closed.*

PROOF. By Proposition 4, $L(S)^\pi=\{\mu\in M(G): \mu * f\in L(S)$ for all $f\in L(S)\}$ so it follows that $L(S)^\pi$ is *so*-closed since $L(S)$ is norm closed. It is clear that $x\in S^\pi$ if and only if $\epsilon_x\in L(S)^\pi$. By Proposition 2 of [6] $x_j\rightarrow x$ if and only if $\epsilon_{x_j}\rightarrow\epsilon_x$ in the *so*-topology. It follows that S^π is closed.

THEOREM 3. *Let S be a subsemigroup of G having nonzero Haar measure. Then $M(S^\pi)\subset L(S)^\pi$ and if $e\in dS$ then $M(S^\pi)=M(dS)=L(S)^\pi$.*

PROOF. Since S^τ is closed, $\mu \in M(S^\tau)$ if and only if $\mathrm{Supp}(\mu) \subset S^\tau$ so it follows from Proposition 2 of [6] that each μ in $M(S^\tau)$ is an *so*-adherence point of the linear span of $\{\epsilon_x : x \in S^\tau\}$. Each ϵ_x in $M(S^\tau)$ is in $L(S)^\tau$ so that $M(S^\tau) \subset L(S)^\tau$ since both $M(S^\tau)$ and $L(S)^\tau$ are *so*-closed (Propositions 2 and 5).

If $e \in dS$ then by Proposition 4, $L(S)^\tau \subset M(dS)$. It follows from Proposition 3 that $M(dS) \subset M(S^\tau)$.

REMARK. T. A. Davis [2] states that $L(S)^\tau \subset M(S^\tau)$ (Theorem 3.5) however it appears that his proof is not complete.

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